c) If $D_1 = D_2$, $\lambda = 1$:

\[(3) \quad X_\lambda = \int d\mu(g) D(g) A D^{-1}(g) = z(\lambda) I .\]

If one takes $A$ such that $\text{tr} A$ is well defined and $\lambda = 0$ (and this is possible e.g. when $A$ has finite rank), one obtains:

\[\text{tr} X_\lambda = \int d\mu(g) \text{tr} A = z(\lambda) d_\lambda ,\]

where $d_\lambda$ is the dimension of $D$.

Now if $(A)$ would be 0, we would have had $X_\lambda = 0$ hence $\text{tr} X_\lambda = \text{tr} A = 0$; which is contradictory. Hence $z(\lambda) = 0$ and $d_\lambda$ is the finite number $\nu / \nu$.

**Theorem 2.** Any irreducible representation of a compact group has finite dimension.

Application to Abelian groups: Take

\[A = D(g_0) , \quad g_0 \in G ; \quad \rightarrow X = D(g_0) = z I ;\]

since the representation is irreducible, it must then have dimension 1.

**Orthogonalization relations.** Since the $D(g)$ have finite dimension, they can be represented by matrices.

Let us choose

\[A = [c_{ik}] = j - \begin{pmatrix} O & 1 \\ -1 & O \end{pmatrix} \]

Then

\[(X_\lambda)_{ij} = \int d\mu(g) D^{(\gamma)}_{\lambda i}(g) D^{(\gamma)}_{\lambda j}(g^{-1}) = \delta_{ij} \delta_{\lambda \lambda} + \frac{1}{d_\delta} \delta_{ik} \delta_{ji} \]

or, since $D(g^{-1}) = D^{-1}(g) = D^*(g) = D_\delta(g)$:

\[(4) \quad \int d\mu(g) D_{\lambda i}^{(\gamma)}(g) D_{\lambda j}^{(\gamma)}(g) = \frac{1}{d_\delta} d_{\lambda i} \delta_{ij} \delta_{\lambda \lambda} .\]

Thus:

**Theorem 3.** The functions $D_{\lambda i}^{(\gamma)}(g)$ form an orthogonal set of functions in $L^2(G)$ (the set of square integrable functions on $G$).

One can prove that $D_{\lambda i}^{(\gamma)}(g)$ form a complete basis of this Hilbert space Peter-Weyl theorem.
One can also prove that \( L^{-x}(g) \) is separable: therefore there is at most a denumerable set of inequivalent irreducible representations.

**Reality condition for irreducible representation.** - Let us now define

\[
Y = \int d\mu(g) D(g) A D^{-1}(g).
\]

If \( D \) is a unitary representation, \( \overline{D} = D^{-1} \) is also a representation. By the same technique we prove \( D^* Y = Y \overline{D} \):

- If \( D \equiv \overline{D} \), \( Y = 0 \).
- If \( D \sim \overline{D} \), \( \exists \ S \rightarrow \overline{D} = S^{-1} D S = D^{-1} \)

\[
Y = \int d\mu(g) D(g) A D^*(g) = \alpha(A) S.
\]

Now,

\[
D = \overline{S^{-1} D S},
\]

\[
D^{* -1} = D = S^2 D S^{-1} S^{-1},
\]

hence \( S S = \lambda I \) (Schur's lemma),

\[
S^{-1} S = \lambda' I \) (Schur's lemma),
\]

i.e. \( S^2 = CS \rightarrow C^2 = 1 \rightarrow S^2 = \pm S \) by convenient choice of a factor. The \( S \)

can be made unitary, hence \( SS = C \).

- If \( D \sim \) to a real representation \( S \) is a multiple of the unit matrix and therefore \( S^2 = +1 \), i.e. \( C = 1 \) (use the fact that \( C \) does not depend on an equivalence). It is easy to prove that this necessary condition is sufficient. Let \( \omega \neq \) all proper values of \( S \) and \( |\omega| = 1 \); then \( D' = (S - \omega) D (S - \omega)^{-1} = D \)

- If \( D \equiv \) to a real representation \( S^2 = -S \).

Thus for any \( A \)

\[
\int d\mu(g) D(g) A^* D^*(g) = \pm \int d\mu(g) D(g) A D^*(g)
\]

**Characters.** - The trace of a representation \( D'(g) \) is called its character \( \chi'(g) = \text{tr} D'(g) \). Since \( \text{tr} D = \text{tr} S D S^{-1} \), the characters are defined on representations up to an equivalence. If two elements \( g, h \) are conjugate \((\exists a \rightarrow g = a h a^{-1})\), then \( \chi'(g) = \chi'(g) \); the character is the same for all members of a conjugate class in the group. One has from (4) with \( i = j, k = l \)

\[
\int d\mu(g) \chi'(g) \overline{\chi'(g)} = \delta_{x}
\]

Furthermore, from (5) with \( A = [e_{ij}] \) one gets:

\[
Y_i = \int d\mu(g) D_{1i}(g) D_{1i}(g) = \pm \int d\mu(g) D_{1i}(g) D_{1i}(g)
\]
Taking the trace: \( (j = k, \ i = k, \ \Sigma_{i,j}) \) one has:

\[
\int d\mu(g) \chi(g^2) = \pm \int d\mu(g) \chi^2(g) = \int d\mu(g) \chi(g) \frac{1}{\chi(g)} = 1
\]

(since \( \overline{D} \sim D \) if \( \chi \neq 0 \)).

**Theorem 4.**

- \( D \sim \overline{D} \) and \( D \sim \) real rep. \( \Leftrightarrow \int d\mu(g) \chi(g^2) = \pm 1 \)
- \( D \not\sim \overline{D} \) and \( D \sim \) real rep. \( \Leftrightarrow \int d\mu(g) \chi(g^2) = -1 \)
- \( D \parallel \overline{D} \) \( \Leftrightarrow \int d\mu(g) \chi(g^2) = 0 \)

**Peter-Weyl Theorem.** - The original proof makes use of the spectral theory of Hermitian operators. We give a simpler proof due to Stone. It is valid only for groups with faithful representations.

**Theorem.** - The functions \( D_{ij}^{\rho}(g) \) of \( g \in \mathcal{G} \) form a complete basis in the space \( L^2(G) \), i.e. if \( \phi(g) \) is orthogonal to \( D_{ij}^{\rho}(g) \) for all \( r, i, j \), it is the zero function. This basis is uniformly complete, i.e. all \( \epsilon > 0 \).

\( \exists \) a finite linear combination \( f_\epsilon \) s.t. \( |f_\epsilon(g) - \phi(g)| < \epsilon \) for any \( g \). In what follows \( D_{ij}^{\rho} \) is assumed to be real; for, if it is not, the representation formed by the direct sum \( D^\rho + \overline{D}^\rho \) is equivalent to a real representation:

\[
S \left( \begin{array}{c|c}
D^{\rho} & D_{ij}^{\rho} \\
\hline
D_{ij}^{\rho} & \overline{D}^{\rho}
\end{array} \right) S^{-1} = \frac{1}{2} \left( \begin{array}{cc}
D + D & i(D - D) \\
-i(D - D)^* & D + D
\end{array} \right)
\]

where

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
i & i \\
-1 & 1
\end{pmatrix}
\]

Let \( D \) be a real faithful representation of \( G \).

Consider the Kronecker products

\( \otimes D \)

for all \( n \)

(note that the Kronecker product of two representations is itself a representation since \( (A \otimes B)(A' \otimes B') = AA' \otimes BB' \)).

We shall prove that the representation \( (\otimes D)_{ij} \) form a complete set, so the same holds for the irreducible representations.

Suppose there exists \( \phi(g) \) orthogonal to all \( (\otimes D)_{ij} \) and different from zero in some neighbourhood of \( g_0 \).
Construct
\[ f(g, N) = A(N) \exp \left( -N \sum \frac{1}{2} (D_u(g) - D_u(g_0))^2 \right), \]
where \( A(N) \) is chosen so that
\[ \int \! d\mu(g) f(g, N) = 1. \]

It is well known that a function of the form of \( f(g, N) \) can be expanded as a power series with an infinite radius of convergence. The terms in this series are evidently linear combinations of the \((\otimes D)_{u1}\).

\[ \lim_{g \to \infty} \int \! d\mu(g) f(g, N) = 1 \]
but \( f(g, \infty) = 0 \) for \( g \neq g_0 \), so that \( f(g, \infty) = \delta(g - g_0) \).

If \( \varphi(g) \) were orthogonal to all \((\otimes D)_{u1}\) then one would have
\[ \lim_{g \to \infty} \int \! d\mu(g) \bar{f}(g, N) \varphi(g) = 0, \]
but
\[ \lim_{g \to \infty} \int \! d\mu(g) \bar{f}(g, N) \varphi(g) = \varphi(g_0), \]
which by hypothesis is different from zero. Thus \((\otimes D)_{u1}\) form a complete set and can be decomposed into a direct sum of irreducible representations which is complete.

This proof shows also that:

1) The set of irreducible representations is at most denumerable.

2) Any faithful representation generates, by \( \otimes \), all the irreducible representations.

Example. – The two-dimensional rotation group: if \( \varphi \) is the angle of rotation \( D^{(2)}(\varphi) = \exp \left( i n \varphi \right) \) \( (n \) integer) \[
\int \! \exp \left( i n \varphi \right) \exp \left( -i m \varphi \right) \frac{d\varphi}{2\pi} = \delta_{nm}. \]

We have therefore proved that for the Hilbert space of square integrable functions defined on the interval \((0, 2\pi)\), the functions \( \exp \left( i n \varphi \right) \) form an orthonormal complete set.
Finite groups. — Let $N_g$ be the number of elements of the group. The complex-valued functions defined on the group form a $N_g$ dimensional linear space (Exercise in GARDING’s note). We have show that the $\sum a^\xi_{\alpha} \Omega_{\alpha}^\xi$-functions are orthonormal. Since they form a complete set we can conclude

$$\sum a^\xi_{\alpha} a_{\alpha}^\xi = N_g.$$ 

The decomposition of a given representation of a compact group into a direct sum of irreducible representations.

From Theorem 1, we can make it unitary.

First case: The representation is finite dimensional.

Then the accident mentioned at the bottom of page 7 will not occur.

The representation is a direct sum of irreducible representations.

Let

$$U(g) = \bigoplus n_\tau D^{(\tau)}(g),$$

where $n_\tau$ is number of times the irreducible representation $D^{(\tau)}(g)$ occurs in $U(g)$. Then

$$\chi(U(g)) = \sum n_\tau \chi^{(\tau)}(g)$$

and from (6)

$$n_\tau = \int d\mu(g) \chi^{(\tau)}(g) \chi(U(g)).$$

If the representation is infinite one has to be more careful.

Let

$$E^{(\tau)}_{ij} = d_{\tau^{(\alpha)}} \int d\mu(g) D^{(\tau)}_{ji}(g) U(g);$$

one can prove

$$E^{(\tau)^*}_{ij} = E^{(\tau)}_{ji}.$$ 

Let

$$E^{(\tau)} = \sum E^{(\tau)}_{ji} = d_{(\alpha)} \int d\mu(g) \chi^{(\tau)}(g) U(g);$$

calculation yields that $E^{(\tau)}$ are the projection operators in Hilbert space onto the subspace $\mathcal{H}^{(\tau)}$ on which act the $\bigoplus$ of representations equivalent to $D^{(\tau)}$ occurring in the direct sum of $U$.

The character function only of the conjugation class $C$ and not of each group element. Let $N_c$ be the number of such classes.
The characters $\chi^\nu \in \mathcal{C}$ span a vector space of at most $N_\nu$ dimensions. Since they are linearly independent (equation (6)) the number $N_\nu$ of irreducible inequivalent representations is $\leq N_\nu$.

**Theorem.** – For finite groups $N_\Phi = N_\nu$.

Consider the $N_\nu$ dimensional linear space of formal linear combinations of group elements. The group multiplication generates naturally a $N_\nu$ dimensional linear representation on this space, the so called regular representation $D_\nu$. It can be shown that $D_\nu$ contains all the irreducible representations, each one $d_\nu(r)$ times. We shall not prove it here.

4. **Structure of the Lorentz group.**

**Notation.** – An event in space time is specified by the four numbers $(ct, \mathbf{r})$. We denote 3-dimensional vectors with an arrow above $\mathbf{r} = (x^i); \ i = 1, 2, 3$. We denote 4-dimensional vectors with a bar below: $\mathbf{x} = (x^\mu); \ \mu = 0, 1, 2, 3$. ($x^0 = ct$).

The separation between two events, $\mathbf{x}, \mathbf{y}$ is given by:

$$(x - y)^2 - (x - y) \cdot (x - y) = (x^0 - y^0)^2 - \sum_{i=1}^{3} (x^i - y^i)^2.$$

We define the metric tensor: $G$

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} G \mathbf{y} = x^\nu y^\nu - \sum_{i=1}^{3} (x^i y^i) = x^\nu g_{\nu \mu} y^\mu; \ \ G = (g_{\nu \mu}) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}$$

We define $g^{\mu \nu}$ as the elements of $G^{-1} = G$:

$$g^{\mu \nu} g_{\nu \sigma} = g^{\mu \sigma} = \delta^\mu_\sigma \quad \text{(The summation convention is understood).}$$

We define the covariant components of a vector

$$x^\mu = g_{\nu \mu} x^\nu.$$

**Definition of Lorentz group.** – The group $\mathcal{L}$ of all linear transformations $\mathbf{x} \to \mathbf{x}'$ such that $(x' - y')^2 = (x - y)^2$. Unless otherwise specified, we shall limit ourselves to the real transformations.

So, we write:

$$x'^\mu = A^\mu_\nu x^\nu + \omega^\mu;$$

$\omega = \{\omega^\mu\}$ is called the translation $rT$;

$A = \{A^\mu_\nu\}$ is called a homogeneous transformation: $rL$. 
The most general Lorentz transformation is called inhomogeneous. The inhomogeneous group will be noted by \( \mathcal{L} \).

Given a homogeneous transformation \( \Lambda \) and a translation \( a \), one denotes the operation \( \mathbf{x} \to \mathbf{x}' = \{x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}\} \) by: \{a, \Lambda\}, being understood that \( \Lambda \) is performed first.

**Multiplication law:**

\[
\{a, \Lambda\} \{b, M\} = \{a + \Lambda b, \Lambda M\}.
\]

**Neutral element (unity):** \{0, 1\},

**Inverse:**

\[
\{a, \Lambda\}^{-1} = \{-\Lambda^{-1}, a \Lambda^{-1}\}
\]

\[
\{a, \Lambda\} = \{a, 1\} \{0, \Lambda\} \quad \text{i.e.} \quad \mathcal{L} = T \cdot L.
\]

Both \( T \) and \( L \) are subgroups of \( \mathcal{L} \).

**Invariant subgroups.** - \( H \) is an invariant subgroup of \( \mathcal{L} \) if for all \( \{a, \Lambda\} \in \mathcal{L} \), \( \{b, M\} \in H \)

\[
\{a, \Lambda\} \{b, M\} \{a, \Lambda\}^{-1} = \{a + \Lambda b - \Lambda M \Lambda^{-1} a, \Lambda M \Lambda^{-1}\} \in H.
\]

One can easily see that \( T \) is an invariant subgroup.

The cosets of \( \mathcal{L} \) modulo \( T \) are:

\[
\{a, \Lambda\} \{b, 1\} = \{a + \Lambda b, \Lambda\}.
\]

The quotient group \( \mathcal{L}/T \) is isomorphic with \( L \).

\( \mathcal{L} = T \cdot L \) but \( \mathcal{L} \neq T \times L \) because, whereas \( T \) is an invariant subgroup, \( L \) is not. \( \mathcal{L} \) is sometimes called the semi direct product of \( T \) and \( L \). The multiplication law is easily remembered by writing in matrix form:

\[
\begin{pmatrix}
1 & 0 \\
a & \Lambda
\end{pmatrix}
\]

(Another illustration of a reducible representation which is not completely reducible). 

— Translation group. \( T \).

**The homogeneous group.** \( L \). Its four pieces. — Since \( \mathcal{L} \) leaves invariant \((\mathbf{x} - \mathbf{y})^{2}\), it leaves invariant \( x^{2} \) (put \( y = 0 \)), \( y^{2} \) (put \( x = 0 \)), and therefore \( \mathbf{x} \cdot \mathbf{y} \):

\[
\mathbf{x} \cdot \mathbf{y} = \Lambda \mathbf{x} \cdot \Lambda \mathbf{y}, \quad \text{i.e.} \quad x^{a} y_{a} = \Lambda^{\mu}_{\nu} x^{\mu} \Lambda_{\sigma}^{\nu} y^{\sigma} g_{\mu \nu}.
\]
as this must be true for all \( x, y \), one has:

\[
\Lambda^\nu_{\gamma} \Lambda^\sigma_{\alpha} g_{\mu\nu} = g_{\alpha\sigma} \quad \text{or} \quad G = \Lambda^\nu G \Lambda^\nu .
\]

This is necessary and sufficient.

The invariance of \( x \cdot x = x^2 \) under \( \Lambda \in L \) allows us to distinguish between

- time-like vectors \( x^2 > 0 \),
- light-like vectors \( x^2 = 0 \),
- space-like vectors \( x^2 < 0 \).

In the affine space (set of points \( 0 + x \)), the sets of points such that \( x^2 = c > 0 \) is the two piece hyperboloid \( \mathcal{H}_u \) \((x^0 > 0)\) and \( \mathcal{H}_l \) \((x^0 < 0)\).

\( x^2 = 0 \) is the light cone \( C_+ (x^0 > 0) \) and \( C_- (x^0 < 0) \);

\( x^2 = c < 0 \) is the one piece hyperboloid \( \mathcal{H}_u \).

Note if \( x^2 = x^2 = c < 0 \) one can go continuously from

\[ x \rightarrow x' . \]

If \( c > 0 \), it is possible only if \( x'^0 x^0 > 0 \).

**Connection of \( L \).** One has \( \det G = \det \Lambda^\nu G \Lambda^\nu \), thus \( \det \Lambda^\nu \Lambda^\nu = (\det \Lambda)^2 = 1 \) so that \( \det \Lambda = \pm 1 \).

One cannot go by a homomorphism from unity \((\det 1 = 1)\) to \( \Lambda \) for which \( \det \Lambda = -1 \); \( L \) is therefore composed of two disconnected parts:

\[
L_+ = \{ \Lambda, \det \Lambda = +1 \} \quad \text{which is the invariant subgroup of proper Lorentz transformations,}
\]

\[
L_- = \{ \Lambda, \det \Lambda = -1 \} \quad \text{which is the set of improper Lorentz transformations (and is not a subgroup).}
\]

\( L \vdash L_- \) consists of two elements.

Now consider real Lorentz transformations: (the complex homogeneous group is denoted by \( CL \)).

One has:

\[
g_{\mu\nu} = \Lambda^\nu_{\alpha} \Lambda^\sigma_{\beta} g_{\mu\nu} = 1 = (\Lambda^\nu_{\alpha})^2 = \sum (\Lambda^\nu_{\alpha})^2 ,
\]

thus

\[
(\Lambda^\nu_{\alpha})^2 = I + \sum (\Lambda^\nu_{\alpha})^2 > 1 .
\]
Thus:
\[ A^0_{\mu} = 1 \quad \{1, 1\}, \quad L^0_{\mu} = 1 \quad \{L^0_{\mu}\} \quad \text{orthochronous group} \]
for the sign of the time component of a true like vector is not changed:
\[ e^{\mu} = A^0_{\mu}x^\mu = A^0_{\mu}x^\mu + A^0_{\mu}x^\mu, \]
\[ \left( \sum_i A^0_{\mu}x^i \right)^2 < \left( \sum_i A^0_{\mu}x^i \right)^2 \]
\[ \sum_i (x^i)^2 = (A^0_{\mu})^2 - 1 \sum_i (x^i)^2 < (A^0_{\mu})^2 (x^0)^2, \]
\[ \text{(*) for a time like vector: } \sum_i (x^i)^2 < (x^0)^2. \]
Thus \[ |A^0_{\mu}x^\mu| > 1 \sum_i A^0_{\mu}x^i, \]
which proves the announced result (for space like vectors \( x^i < 0 \), the proof does not go through). Similarly if \( A^0_{\mu} < -1 \) the time component of a time-like vector reverses sign under \( L \).

Thus we have found in \( L \) four disconnected sets of transformations:
\( L^+ \), \( L^- \), \( L^+ \), \( L^- \). Correspondingly \( \mathcal{L} \) has four disconnected pieces.

**Decomposition of Lorentz transformations into plane reflections.**

**Def.** The reflection \( \Sigma_n \) through a plane orthogonal to \( n \) is defined by
\[ x' = x - 2\frac{n \cdot x}{n^2} n. \]

Note that \( \Sigma_n = \Sigma_{-n} \) and \( \Sigma_n n = -n \).

**Exercise 1.** Show that \( A\Sigma_n A^{-1} = \Sigma_n \) with \( n' = A_n \) (\( \Sigma_n \) is explicitly known, \( A^{-1} = G A^T G \)).

**Exercise 2.** If \( n_i \) are linearly independent and \( A = \Sigma_{n_i} \Sigma_{n_j} \Sigma_{n_k} \),
\[ 4p = p \leftrightarrow a_i \cdot p = 0 \quad \text{for all } a_i \quad (\approx \text{evident}); \]
\( A \) is of the form
\[ A_{\mu\nu} = q_{\mu\nu} + \sum_i c_i a_i a_i(p), \quad \text{with } c_{\mu\nu} = -2(n_i). \]

and
\[ 4p' = p' + \sum_i a_i a_i(p). \]
hence
\[ \lambda_i = 0; \quad \lambda_i = \sum \epsilon_{ij} u_j \cdot p; \]
for \( i = k, n_k \cdot p = 0 \) then \( i = k - 1 \Rightarrow n_{k-1} \cdot p = 0 \), then......

**Theorem.** Any \( \Lambda \in L \) can be written as a product of at most 4 plane reflections. (This is a particular case of a stronger theorem: \( \Lambda \) a rotation in \( k \)-dimensional space, i.e. a linear transformation leaving invariant the non-degenerate symmetric form
\[ \sum_{i=1}^{k} g_{ij} x^i y^j \]
can be written as a product of at most \( k \) plane reflections). The proof goes by induction on \( k \).

We have three cases.

1) \( \exists x \) such \( \Lambda x = x \) and \( x^2 \neq 0 \). Then the \( k - 1 \) dimensional space \( \mathcal{E}_{k-1} + x \) is left invariant by \( \Lambda \): \( \Lambda \mathcal{E}_{k-1} = \mathcal{E}_{k-1} \) and \( \Lambda \) is at most the product of \( k - 1 \) symmetries;

2) no invariant \( x \) but \( \exists y \) such that \((\Lambda y - y)^2 \neq 0\). The symmetry \( \Sigma_a \) (where \( a = \Lambda y - y \) ) exchanges \( \Lambda y \) and \( y \). Hence \( \Sigma_a \Lambda \) leaves \( y \) invariant, i.e. belongs to first case;

3) for all \( x \), \( a = \Lambda x - x \) is a light vector,
\[ a^2 = 0. \]

Part of this proof is more difficult in the general case, but in the case of real Lorentz group in 4 dimension it is very easy to prove that \( \Lambda \) is then the identity.

The symmetries and the 4 pieces.

\[
(\Sigma_n^a)_{ij} = q_{ij} - 2 \frac{n^i n^j}{n^2}, \quad \Sigma_n^2 = 1, \quad \text{Tr} \, \Sigma_n = 2; \quad \det \, \Sigma_n = -1
\]

\[
(\Sigma_n^a)^2 = 1 - 2 \frac{n^i n^j}{n^2}, \quad \text{i.e.} \quad (\Sigma_n^a)^2 - 1 = 2 \frac{(n^a)^2}{n^2}, \text{ the sign of which is } -n^2.
\]

Hence \( \Sigma_n \in L \); if \( n \) time like \( \Sigma_n \in L^\perp \),
\( n \) space like \( \Sigma_n \in L^\perp \).

Hence, elements of \( L^\perp \) – product of even (2, 4) number of \( \Sigma_n \) with even\((0, 2, 4)\) number of time-like \( n \)
$L_+^b = \text{product of even number of } \Sigma_n \text{ with odd (1, 3)}$  
$\text{number of time-like } n$

$L_+^a = \text{product of odd (1, 3) number of } \Sigma_n \text{ with even number of time-like } n$

$L_+^e = \text{product of odd number of } \Sigma_n \text{ with odd number of time-like } n$

$L_+^c$ is connected:

$\forall \Lambda \in L_+^c, \quad \Lambda = \Sigma_n \Sigma_n \Sigma_n \Sigma_n.$

Since the sign of $n$ is arbitrary, for time-like vector $n$ choose the time component $> 0$. Now it is possible to vary continuously the time-like vector to a fixed time-like $t$; the space-like, to a fixed $s$; since $(\Sigma_n)^2 = 1$, one has varied continuously $\Lambda \rightarrow I$ (the identity), except in the case where the situation was for instance $\Lambda = \Sigma_s \Sigma_s \Sigma_s \Sigma_t$, where $s' = \Sigma_s t$ (from Exercise 2, p. 000), then when $t' \rightarrow t$, $\Lambda \rightarrow I$.

Transitivity of $L_+^c$ on $\mathcal{H}_{+\pm}$, $\mathcal{H}_{-\pm}$, $\mathcal{C}_+$, $\mathcal{C}_-$, $\mathcal{H}_i$.
Notation defined on pages 11-12.

Remark. If $p'^2 - p^2$ and $(p' + p)^2 \neq 0$, $\Sigma_{p' - p} (-p) = p'$.

Hence if $(p' - p)^2 \neq 0$, $\Sigma_{p' - p} p = p'$.

We define

$S_{p', p} = \Sigma_{p' - p}.$

Now we can easily solve the problem:
Given $p$ and $p'$ ($p'^2 - p^2$ in the same $\mathcal{H}_{+\pm}$ (respectively $\mathcal{H}_{-\pm}$, ...), find $S \in L_+^c$ such that $p' = Sp$.

Answer:

1) for $\mathcal{H}_{+\pm}$ (respectively $\mathcal{H}_{-\pm}$), $S_{p', p}$ (for $p + p' \in \mathcal{H}_{+\pm}$), hence $S_{p', p} \in L_+^c$;

2) for $\mathcal{C}_+$ (respectively $\mathcal{C}_-$):

If $p''$, $p$ are linearly independent $(p' - p)^2 < 0$ let $s$ such that $s^2 < 0, n \cdot p = n \cdot p' = 0; \Sigma_{p' - p} \Sigma_n$ is a solution.

If $p' = xp$, take $t$ time-like ($\Rightarrow t: p \neq 0$) call $p' = \Sigma_t p$ then $\Sigma_{p' - p} \Sigma_t$ is a solution;

3) for $\mathcal{H}_s (p + p')^2 + (p - p')^2 = 4p^2 - 4p'^2 < 0$, hence at least one of $p + p'$ or $p - p'$ is space-like.

If $p + p'$ space-like $S_{p'}$ is solution $p + p'$ time-like $\Sigma_{p' - p} \Omega$ is solution ($s^2 < 0$, $s \cdot p = s \cdot p' = 0$).
Little group of vector. - Wigner called little group of \( p = L \), the set of \( \Lambda \) such that \( \Lambda p = p \).

\( p \) time like from Exercise 2, page 12. \( L \) is generated by \( \Sigma_n \) with \( n \cdot p = 0 \), i.e. \( n \) space like. Hence \( L \) isomorphic to \( O_3 \), the orthogonal group into 3 dimensions. Its connected part is \( O^*_3 \), i.e. the group of rotations into 3 dimensions.

It can also be seen this way: \( L \) and \( \Lambda \Lambda L = \Lambda^{-1} \) are isomorphic groups for a fixed \( \Lambda_0 \). Choose \( \Lambda_0 \) such that \( \Lambda^{-1} p \) is on the time axis (if \( p \) is the four momentum, \( \Lambda^{-1} p \) brings the particle at rest.

\( p \) space like similar argumentation yields Lorentz group on space with one time axis, 2 space axis.

\( p \) light like the space orthogonal to \( p \) contains \( p \): \( p^2 = 0 \). Let \( n_1, n_2 \) transverse vectors orthogonal to \( p \), i.e.

\[ n_1 \cdot p = n_1 \cdot n_2 = 0 \]

and \( n_i = (0, n_i) \) hence \( n_1 \cdot p = n_2 \cdot p = n_1 \cdot n_2 = 0 \)

we take the \( n \) unitary

\[ n_1^2 = -n_2^2 = n_2^2 = -n_3^2 = -1 \]

Call

\[ n' = \cos \frac{\theta}{2} n_1 + \sin \frac{\theta}{2} n_2 \]

\[ n' \cdot p = 0, \quad n^2 = -1 \]

The most general unit vector orthogonal to \( p \) is of the form (\( x \) real arbitrary):

\[ n' + p, \quad \text{indeed} \quad (n' + x p) \cdot p = 0, \quad (n' + x p)^2 = -1 \]

The elements of the little group of \( p \) are therefore the product of at most 3 arbitrary symmetries of the type \( \Sigma_{n + x p} \).

(Exercise 2, page 12).

Connected little group. Its elements are the product of 2 symmetries

\[ \Sigma_{n + x p} \Sigma_{n + x' p} \]

Let us call \( R(\theta) = \Sigma_{n} \Sigma_{n} \); it is a rotation around \( p \) of angle \( \theta \); so is

\[ \Sigma_{n} \Sigma_{n} = R(\theta^2 - \theta) \]

and call

\[ T'(x') = \Sigma_{n} \Sigma_{n} \Sigma_{n}, \quad \text{example} \quad T(x_1) T(x_2) \]

note \( \Sigma_{n} \Sigma_{n - x p} = \Sigma_{n - x p} \Sigma_{n} \) (see pages 12-13).
Any element of the connected little group of $p$ can be written:

$$
\Sigma_{n}^{r} + \Sigma_{m}^{s} \Sigma_{p}^{t} = \Sigma_{n}^{r} + \Sigma_{m}^{s} \Sigma_{p}^{t} \Sigma_{p}^{t} - F(x',0',0') T(x)
$$

We can generate the whole group from

$$
R(\theta) T_{1}(x_{1}) T_{2}(x_{2})
$$

then $\theta$, $x_{1}$, $x_{2}$ are the three parameters of the group.

From $\Lambda \Sigma_{n} \Lambda^{-1} = \Sigma_{n}$ that this group is just isomorphic to the 2 dimensional connected euclidean group, i.e. the group of translation $t = x_{1} t_{1} + x_{2} t_{2}$ and rotation around the origin of angle $\theta$. Indeed the group law is similar to that of the inhomogeneous Lorentz group and can be represented by the matrice

$$
\begin{pmatrix}
1 & 0 & 0 \\
x_{1} & \cos \theta & -\sin \theta \\
x_{2} & \sin \theta & \cos \theta
\end{pmatrix}
$$

The isomorphism can be extended to the symmetries. $\Sigma_{n}$ corresponds to the symmetry defined by $t_{1}$, i.e. through $t_{1}$ ($\perp t_{1}$).

**Summary.** - Little group of $p$:

- $p$ time like: isomorphic to 3-dimensional rotation group,
- $p$ light like: isomorphic to 2-dimensional euclidean group,
- $p$ space like: isomorphic to 3-dimensional Lorentz group.

The covering group $U_{3}$ of $O_{v}^{(3)}$ (rotations in 3 dimensions). - We use the 3 Pauli matrices $\tau = \{\tau_{i}\}$ $i = 1, 2, 3$ such that $\tau_{i}^{3} = \tau_{i}$;

$$
(30)
\tau_{i} \tau_{j} + \tau_{j} \tau_{i} = 2 \delta_{ij}
$$

To each 3-vector $\mathbf{x}$, we associate the matrix $x = \sum_{i} x_{i} \tau_{i} = \mathbf{x} \cdot \tau$. Note that

$$
(31)
\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}(\mathbf{xy} + \mathbf{yx}) ; \quad x^{2} = x \cdot x
$$

if $\mathbf{x}' = \Sigma_{n} \mathbf{x}$ we have

$$
(32)
\mathbf{x}' = \mathbf{x} - (\mathbf{n} \cdot \mathbf{x}) \mathbf{n}^{-2} \mathbf{n} = - \mathbf{n} \mathbf{x} \mathbf{n}^{-1}
$$
Let $r_{21} = r_2 r_1$; it corresponds to the rotation

$$R_{21} = \Sigma_{n_3} \Sigma_{n_1}, \quad \text{if} \quad y = R_{22} x$$

one has

$$y = r_{21} x r_{21}^{-1} \quad \text{with} \quad r_{21} = n_2 n_1.$$  

From this it is easy to conclude that the matrices $r$ multiply as do the rotations, i.e.

$$R \rightarrow r, \quad R' \rightarrow r', \quad R'' = R R' \rightarrow r r' = r''.$$  

This correspondence is however up to a sign, because $\Sigma_n$ and $\Sigma_{-n}$ are the same symmetry; hence both $\pm r$ correspond to $R$.

If the rotation is defined by $n, \omega$, axis and angle of rotation $r(n, \omega)$ can easily be computed (use $ab = a \cdot b + i(a \cdot b) \cdot \tau$). We obtain taking $n^2 = 1$

$$r(n, \omega) = \pm \left( \cos \frac{\omega}{2} - i \sin \frac{\omega}{2} \right) = \pm \exp \left[ i \frac{\omega}{2} n \right] = \pm u(\omega, n),$$

**Remarks:**

a) The matrices $u(\omega, n) = \exp \left[ -(i\omega/2)n \right]$ for all values of $n$ (with $n^2 = 1$) and $\omega$ generates a group, $U_2$ the unitary unimodular group.

Indeed

$$\omega_1, n_x, n_y, n_z \text{ real } \Rightarrow \left( \frac{\omega}{2} n \right)^* = \frac{\omega}{2} n \Rightarrow u \text{ unitary}$$

$$\text{tr } n = 0 \Rightarrow \text{det } u = 1.$$  

b) Conversely a unitary 2 by 2 matrix $u$ can be written $u = \exp \left[ -i h \right]$ where $h$ is a 2 by 2 hermitian matrix and any $h$ can be written $(\omega/2)n$.

c) What we have found therefore is not a true representation of the rotation group $O^+_3$ but a representation up to a sign only.

We also prove $U_2 \cong O^+_3$, where the homomorphism $f$ is a two to one correspondence. Kernel of $f$ is $u = 1$ and $u = -1$ (which form the two element group $Z_2$).

$$u(\omega, n) \rightarrow r(\omega, n).$$

d) Note that $u(2\pi, n) = -1$ whatever $n$.

**Definition of Poincaré group, $\pi_1$.** - For a topological group consider continuous mapping $s_1$ of the circle $S_1$ (= sphere in 1 dimension; we can generalize to $S_n$) into the group manifold and such that this mapping contains the
identity. (By translation on the group any other fixed point can be chosen). If a given \( s_i \) called \( s'_i \) can be, by continuous transformation, transformed into \( s''_i \) (another \( s_i \)), then \( s_i \) and \( s'_i \) are said «homotopic».

It is easy to see that «homotopy» is an equivalence relation among the \( s_i \). We can define the «product» of two \( s_i \), when \( S_i \) is oriented. This is a composition law for the equivalence class, which gives a group structure. (the identity class is that of the \( s_i \), which, by continuous deformation can be reduced to a point).

This group is called \( \pi_i \) or Poincaré group.

**Simply connected group.** — It is a group whose Poincaré group has only one element. All closed paths in the group are homotopic. Example, \( U_z \) is simply connected.

The most general matrix of \( U_z \) is of the form

\[
\begin{pmatrix}
\alpha & \beta \\
\beta & \alpha
\end{pmatrix}
\]

with \( \alpha \bar{\alpha} + \beta \bar{\beta} = 1 \). If \( \alpha = x_1 + ix_2 \), \( \beta = \beta_1 + i\beta_2 \), the condition is \( x_1^2 + x_2^2 + \beta_1^2 + \beta_2^2 = 1 \). This is the equation of \( S_2 \), the 3 dimensional sphere. Consider a closed path in it, passing through the point \( 1, 0, 0, 0 \) (unit of the group). By continuous deformation this path can be made «plane» (i.e. contained in a 3 dimensional plane) and this plane can be moved continuously to the tangent plane in \( 1, 0, 0, 0 \).

**Covering group \( C \) of \( G \).** — It can be shown (see C. J. PONTRJAGIN: *Topological Groups*, § 47, for the proof and the precise sufficient conditions) that for any Lie group \( G \), there exists a unique simply connected group \( C \), locally isomorphic to \( G \), homomorphic to \( G \). Then, the kernel \( \pi_1 \) of the homomorphism is isomorphic to the Poincaré group of \( G \). This unique \( C \) is called the covering group of \( G \). We have therefore proved that \( U_z \) is the covering group of \( O_z^+ \). We also know the two homotopy class of paths of \( O_z^+ \). The identity class is that of closed paths which are image (by the homomorphism \( U \rightarrow O_z^+ \)) of closed path of \( U \). They can be shrunk to a point since \( U \) is simply connected. The closed paths in \( O_z^+ \) which are image of continuous path from 1 to \(-1\) in \( U \) cannot be shrunk to a point. A way to see the topology of \( O_z^+ \) is to plot each rotation \( n, \omega \) at the tip of the vector \( \omega n \) with \( 0 < \omega < \pi \) and identify the points \( \pm \omega n \); closed paths which contain an odd number of such points \( \pm \omega n \) cannot be homotopic to zero.

**The covering group \( C \) of \( L_z^+ \).** — What we have done for \( O_z^+ \) with the Pauli matrices can be done now for the Lorentz group \( L \) and the Dirac matrices.

\[(\gamma^\alpha, i\gamma^\alpha) = 2g^{\alpha\beta} \quad \text{when} \quad [A, B] = AB + BA \]
We write \( i\gamma^{\mu} \) for the matrices, because it is possible to choose the \( i\gamma^{\mu} \) real. Indeed the \( 4\gamma^{\mu} \) generate an algebra of 16 linearly independent matrices

\[
\gamma^{(a)} = 1, \quad \gamma^{\mu}, \quad i\sigma^{\mu\nu} = \frac{1}{2} [\gamma^{\mu}, \gamma^{\nu}], \quad \gamma^{5}\gamma^{\mu}, \quad \gamma^{5} = \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}.
\]

The 32 matrices \( \pm \gamma^{(a)} \) form a group of elements \( \gamma^{(a)} \). We can compute (see pages 555, 666)

\[
(35) \quad \frac{1}{32} \sum_{a} \text{Tr} (\gamma^{(a)}\gamma^{5}) = \frac{1}{16} \sum_{a} \text{Tr} (\gamma^{(a)})^2 = 1.
\]

We define the correspondence \( x \to x \).

Equations similar to (31), (32), (33) hold.

To summarize, \( A \in L^1 \) can be decomposed into (see page 14)

\[
A = \sum_{n_1} \sum_{n_2} \sum_{n_3} \sum_{n_4} a_{n_1} \quad \text{with} \quad |a_{n}^2| = 1.
\]

The correspondence \( A \to \pm n_1 n_2 n_3 n_4 = S(A) \) is a representation up to a sign of \( L^1 \).

This representation is reducible

\[
i\gamma^{5}S(A) = S(A)i\gamma^{5}.
\]

Since \((i\gamma^{5})^2 = 1\) and \(\text{tr} \gamma^{5} = 0\), \(\gamma^{5}\) can be written

\[
\gamma^{5} = (1 \otimes 1) + (1 \otimes 1).
\]

and one finds easily that the real four dimensional representation \( S(A) \) breaks into the direct sum of two charge conjugated two dimensional representations: e.g. take

\[
\gamma^{\alpha} = (\tau_1 \otimes 1) ; \quad \gamma^{\beta} = (i\tau_2 \otimes 1).
\]

Another method to prove it directly is:

We form the matrices \( X \) for a given \( x \) according to

\[
X = x^\alpha 1 - \sum_i x_i \tau_i,
\]

\[
x : \quad X = \begin{pmatrix}
x^0 + x^3 & x^1 - i x^2 \\
-x^1 + i x^2 & x^0 - x^3
\end{pmatrix}
\]

\[
(36)
\]
and notice that \( \det X = x^2 \).

Corresponding to \( x' = A x \) we write

\[
X'^\mu = A_{\mu \nu}^\rho x^\rho .
\]

For a light-like vector \( x \) \((x^2 = 0)\) this has the form

\[
\gamma^\nu \gamma^\rho = \Lambda_{\nu \sigma}^\rho \delta^\sigma_\rho .
\]

In this case one finds that it is possible to write

\[
(37) \quad X' = A X B^\rho \quad \text{or} \quad X' = A X^\rho B^\rho ,
\]

where the requirement \( \det X' = \det X \) imposes the condition \( \det A \cdot \det B = 1 \) which leaves a great choice of possible \( A, B \). If we suppose that \( \det A = \det B = 1 \) we still have an ambiguity in sign: we can choose either \( A \) and \( B \) or \(-A \) and \(-B \).

The real homogeneous connected Lorentz group. - For this group \( x \) is real and \( X \) hermitian

\[
(39) \quad X^* = X, \quad X'^* = X' .
\]

So that

\[
(B^\rho)^* X A^* = A X B^\rho .
\]

or equivalently

\[
X A^* (B^\rho)^{-1} = (B^\rho)^{-1} A X .
\]

Writing

\[
X A^* (B^\rho)^{-1} = [A^* (B^\rho)^{-1}]^* X ,
\]

we have

\[
F = A^* (B^\rho)^{-1}
\]

\[
X F = F^* X ;
\]

this holds for all hermitian matrices; in particular for \( X = 1 \) so that

\[
F = F^* .
\]

By Schur's Lemma \( \exists \) a scalar \( \lambda \) such that

\[
F = \lambda I .
\]
so that:

\[ A^* = 2B^2 : \]

but \( \det A = \det B \) so that:

\[ A^* = \pm B^* . \]

For the group \( L_+^1 : \)

\[ X' = A X A^* ; \]

for the set (not a group!) \( L_+^1 : X' = -AXA^* . \)

Conversely, take an \( A \) such that \( \det A = 1 \); the correspondence \( X \rightarrow X' = AXA \) is a linear correspondence \( x \rightarrow x' \) preserving \( x^2 \).

Thus we have a homomorphism between the Unimodular group \( G_2 \), i.e. the group of all \( 2 \times 2 \) matrices of determinant 1, and \( L_+^1 : \)

\[ \Lambda \in L_+^1 \rightarrow \pm \Lambda n G_2 . \]

**Examples.** - Rotation:

\[ (n, \omega), \quad A \text{ unitary } U = (U^*)^{-1} = \pm \left( \cos \frac{\omega}{2} - i n \cdot \tau \sin \frac{\omega}{2} \right) . \]

« Pure » Lorentz transformation:

\[ (I, \beta), \quad A \text{ hermitian positive definite } = H = H^* = \pm \left( \cosh \frac{\beta}{2} + I \cdot \tau \sinh \frac{\beta}{2} \right) \]

(a change in « velocity » \( \beta \) (rel. \( \beta = v/c \) in the \( I \) direction).

Now, any matrix can be written as the product of a unitary matrix \( U \) and a hermitian positive definite matrix \( H : \)

\[ A = U H : \]

\( AA^* \) is a positive definite hermitian matrix; \( AA^* = H^2 \rightarrow 3 \) a unique positive definite square root of \( H^2 ; \) \( H \cdot H^{-1}A \cdot A^*H^{-1} = (H^{-1}A)(H^{-1}A)^* = 1 \); since \( H^{-1}A \) exists \( H^{-1}A \) is unitary; \( A = HU \).

Thus every Lorentz transformation has a unique decomposition into the product of a rotation and a pure Lorentz transformation.

We have also explicitly displayed the 6 parameters of the \( L_+^1 \). Note that if we consider « imaginary rotations » \( n, i \omega \) one just obtains the « pure » Lorentz transformation \( I = n, \beta = \omega \) (see (41) and (41')).

This shows that \( L_+^1 \) is isomorphic to the complex rotation group \( CO_3^{(+)} \) in the same way \( C_2 \) is the complex group of \( U_2 \). We leave to the reader to prove that \( C_2 \) is the covering group of \( L_+^1 \) with \( C_2/Z_2 = L_+^1 . \)
Complex Lorentz group CL. — If we do not add condition (39) to the conditions (37) and (38) we obtain a representation up to a sign of the complex Lorentz group $\mathcal{C}L$ (= complex connected orthogonal group $CO_4^+$), i.e. the group, for which $x^4$ is invariant for all $x$ with complex co-ordinates. Any $\in \mathcal{C}L$ is represented, up to a sign, by a couple of unimodular $2 \times 2$ matrices $A$ and $B$.

Hence the direct product $\mathcal{C}_2 \times \mathcal{C}_2$ is a representation up to a sign of $\mathcal{C}L$, i.e.

$$\mathcal{C}L = \mathcal{C}_2 \times \mathcal{C}_2 / \mathbb{Z}_2.$$

Lie algebra of the inhomogeneous Lorentz group. — As we saw, a general element of the group may be written

$$A = \begin{pmatrix} 1 & 0 \\ a & A \end{pmatrix} = A(... \alpha^i ...),$$

where $\alpha^i$ are the parameters labelling the elements and for the identity

$$I = A(0, 0, ...).$$

We construct the infinitesimal operators in the neighbourhood of the identity defined by (see Prof. Racah's lecture):

$$D_i = \left( \frac{\partial A}{\partial x^i} \right)_{\alpha^i = 0}.$$

For a one parameter abelian group (if it exists) we have

$$A(x^i_1 + x^i_2) = A(x^i_1)A(x^i_2),$$

where $a^i_1, x^i_2$ are two values of the same parameter.

We solve this functional equation by differentiating with respect to $x^i_1$ holding $x^i_2$ fixed, and then putting $x^i_1 = 0$:

$$A'_i(x^i_2) = D_i A(x^i_2).$$

Thus we have

$$A(x^i) = \exp [x D_i].$$

We remark that since the exponential of a matrix is well defined there is no difficulty when the operators are matrices; it should be remembered that if

$$[D_1, D_2] \neq 0, \quad \exp [D_1 + D_2] \neq \exp [D_1] \cdot \exp [D_2].$$
Translation group. — For an infinitesimal translation \( A(a, 1) \approx 1 + D_\mu a^\mu \), where \( a^\mu \) are the parameters and \( D_\mu \) are the four corresponding infinitesimal operators. For a finite translation we obtain

\[
U(a, 1) = \exp[iD_\mu a^\mu] = \exp[iP_\mu a^\mu],
\]

where we have introduced

\[
P_\mu = -iD_\mu.
\]

Physicists introduce \( P_\mu \), because when \( U \) is unitary, \( P_\mu \) are hermitian and are observables. Indeed they correspond to energy and momentum. The transformation properties of \( P_\mu \) and \( L^\lambda \) can be obtained by the use of (17):

\[
U(0, 1)U(a, 1)U(0, 1)^{-1} = U(\lambda a, 1)
\]

and (16); one obtains

\[
U(0, E)P_\mu U(0, 1)^{-1} = A_\mu P_\mu.
\]

Furthermore the \( P_\mu \) commute hence

\[
[P_\mu, P_\nu] = 0.
\]

Homogeneous group

\[
A G.1 = G.
\]

The treatment is very similar to that of the orthogonal rotation group.

\( \lambda A \in O_{\mu}, \quad A^T A = 1. \)

Differentiation with respect to a parameter yields (by putting the parameter equal to zero)

\[
\frac{\partial A}{\partial \lambda} \bigg|_{\lambda=0} + 1 A^T \bigg|_{\lambda=0} = 0,
\]

\[
D^T + D = 0.
\]

For the homogeneous Lorentz group we have form (17)

\[
\mathcal{N}^T G + G \mathcal{M} = 0
\]

putting

\[
D = G \mathcal{M}.
\]
We have \( D^i + D_i = 0 \), i.e. \( D_i \) is skew-symmetric. It is \( 4 \times 4 \) so there are six parameters.

We define the matrices \( e_{\alpha \beta} \) as before

\[
(e_{\alpha \beta})_{\mu} = \delta_{\alpha \mu} \delta_{\beta \nu},
\]

that is

\[
e_{\alpha \beta} e_{\beta \gamma} = e_{\alpha \eta} e_{\eta \delta}.
\]

These \( e_{\alpha \beta} \) provide us with a convenient representation of the \( D_{\mu \nu} \) (infinitesimal operator of the rotation in the 2-plane \( \mu - \nu \))

\[
D_{\mu \nu} = e_{\mu \nu} - e_{\nu \mu}.
\]

We have

\[
D_{\mu \rho} D_{\alpha \beta} = (e_{\mu \nu} - e_{\nu \mu})(e_{\alpha \eta} - e_{\eta \alpha})
= e_{\mu \nu} e_{\alpha \eta} - e_{\mu \nu} e_{\eta \alpha} - e_{\nu \mu} e_{\alpha \eta} + e_{\nu \mu} e_{\eta \alpha}
= e_{\mu \alpha} \delta_{\nu \eta} + e_{\nu \alpha} \delta_{\mu \eta} - e_{\nu \alpha} \delta_{\mu \eta} - e_{\mu \alpha} \delta_{\nu \eta}
\]

hence

\[
[D_{\mu \nu}, D_{\alpha \beta}] = D_{\mu \alpha} \delta_{\nu \beta} + D_{\nu \alpha} \delta_{\mu \beta} - D_{\nu \alpha} \delta_{\mu \beta} - D_{\mu \alpha} \delta_{\nu \beta}.
\]

Introducing

\[
\tilde{M} = i\tilde{M} = i\mathcal{Q}D,
\]

\[
[M_{\mu \nu}, M_{\alpha \beta}] = \tilde{i}[g_{\mu \alpha} M_{\nu \beta} + g_{\mu \beta} M_{\nu \alpha} - g_{\mu \beta} M_{\nu \alpha} - g_{\mu \alpha} M_{\nu \beta}]
\]

since the homogeneous Lorentz group is semi-simple we may find the Casimir operators in a straightforward way (see Prof. Racah lecture).

So far we have only in (55) the Lie algebra of \( L^+ \). That is, given any two elements, we know their commutator, not their products.

**Enveloping algebra of a Lie algebra.** - Example \( E \) and \( \mathfrak{g} \).

This consists of all possible formal polynomials formed from the elements of the Lie algebra.

**Centre of the enveloping algebra.** - The centre \( C \) of the enveloping algebra \( E \) is the set of all elements \( c \in E \) which commute with every other element:

\[
C = \{ c \in E | \forall j \in E : jc = c j \}.
\]
Centre of the enveloping algebra of the inhomogeneous group. — It is convenient to define the 3-dimensional "pseudovector"

\[
(J^1, J^2, J^3) = (M^{31}, M^{12}, M^{23}),
\]

and the 3-vector

\[
(N^1, N^2, N^3) = (M^{01}, M^{02}, M^{03}).
\]

We have from (55)

\[
[J^1, J^2] = i \epsilon^{ijk} J^k,
\]

which physicists write symbolically

\[
J \wedge J = iJ.
\]

We also obtain

\[
\mathbf{N} \wedge \mathbf{N} = -iJ,
\]

\[
[J_\nu, N_\mu] = i \epsilon^{\nu\mu\rho} N_\rho = [N_\nu, J_\mu].
\]

As is well known

\[
[J^2, J^1] = 0,
\]

but

\[
[J^3, N] \neq 0, \quad [N^2, N] \neq 0.
\]

For the homogeneous group the invariants (i.e., elements of the center) are

\[
J^1 = \frac{1}{2} M_{\mu \nu} M_{\nu \rho}, \quad J \cdot \mathbf{N} = \frac{i}{2} \epsilon^{\mu \nu \rho} M_{\mu \nu} M_{\nu \rho} = \det M_{\mu \nu},
\]

They do not, however, commute with the \(P\)'s and so do not belong to the Centre for the inhomogeneous group.

For the inhomogeneous group we have to include the elements \(P_\mu\) of the translation group. We find

\[
[P_\mu, J_\rho] = i(g_{\mu \alpha} P_\alpha - g_{\mu \beta} P_\beta)
\]

so that using \([A, B, C] = [A, B]C + [A, C]B\) we find that: \(P_\mu = P^\mu P_\mu\) commutes with the \(M_{\mu \nu}\).

It also commutes with the \(J\), so that it is one element of the centre. The other invariant was found by Pauli; let

\[
W = \frac{1}{2} \epsilon^{\rho \sigma \delta} P_\rho M_\sigma.
\]
GROUP THEORY

then

\[ P_\lambda W^\lambda = 0 \]

and

\[ [P_\mu, W_\lambda] = 0. \]

We compute

\[ [W_\lambda, M_{\mu\nu}] = i(g_{\lambda\nu} W_\mu - g_{\lambda\mu} W_\nu), \]
\[ [W_\lambda, W_\mu] = i\epsilon_{\lambda\mu\nu\rho} P^\nu W_\rho, \]

(note that \( \epsilon_{\lambda\mu\nu\rho} = -\epsilon_{\mu\lambda\nu\rho} \) since \( \det g = -1; \epsilon_{0123} = 1 \)).

We see that

\[ W^2 = W_\rho W^\rho \]

is also an element of the centre of the enveloping algebra.

Summary. - Invariants:

for \( \mathcal{E} \): \( P^2, \quad W^2, \)

for \( \mathcal{L} \): \( J^2 - N^2, \quad J \cdot N. \)