A method is developed for constructing single valued rational 4-point functions of primary fields for su\(_2\) conformal current algebra satisfying the Knizhnik-Zamolodchikov equation. For integer conformal dimensions \(\Delta\) these rational solutions are proven to be in one-to-one correspondence with non-diagonal modular invariant partition functions of the D-even and E-even series of the ADE classification.

In Memory of Nikolai Nikolaevich Bogolyubov and Mikhail Constantinovich Polivanov

1. INTRODUCTION

Axiomatic local quantum field theory methods developed, in particular, in the work of N. N. Bogolubov, B. V. Medvedev, M. C. Polivanov and their students are becoming instrumental in the study of 2-dimensional (2D) conformal field theory (CFT). The “algebraic local quantum theory” approach [1] to classifying 2D CFTs associated with a given chiral current algebra proceeds as follows [2]. One begins with an algebra \(\mathcal{A}\) of single-valued local Bose fields on the circle. (The circle \(z \bar{z} = 1\) appears as a compactified light ray of 2-dimensional Minkowski space.) The elements of \(\mathcal{A}\) are viewed as operators in the vacuum Hilbert space \(\mathcal{H}_0\) which carries a unitary positive energy representation of the Möbius group \(SU(1,1)\) such that each energy eigenvalue appears with a finite multiplicity and there is a unique Möbius invariant vacuum state. \(\mathcal{A}\) contains the rightmovers’ component of the stress energy tensor

\[
T(z) = \frac{1}{4}(T_0^0 - T_0^1 + T_0^1 - T_0^0) = \sum_{n \in \mathbb{Z}} L_{n-2}z^{-n}
\]  

(1.1)

where \(L_0\) and \(L_{\pm 1}\) are the \(su(1,1)\) generators, \(L_n^* = L_{-n}\). According to the Lüscher-Mack theorem (see, e.g., [3], or Sec. 3A of [4]) \(T\) gives rise to a (unitary, positive energy) representation of the Virasoro algebra with a fixed central charge \(c\). The vacuum expectation values of products of “chiral currents” which generate \(\mathcal{A}\) admit analytic continuation as rational (meromorphic) functions on the Riemann sphere. We are interested in local (2-dimensional) CFT whose observable algebra is an extension of \(\mathcal{A} \oplus \overline{\mathcal{A}}\); here \(\overline{\mathcal{A}}\) is the antianalytic (leftmovers’) counterpart of the “analytic” (rightmovers’) algebra \(\mathcal{A}\). (We only consider isomorphic chiral algebras \(\mathcal{A}\) and \(\overline{\mathcal{A}}\) corresponding to the same value of \(c\).) The first step in the CFT program requires constructing the positive energy representations of \(\mathcal{A}\) generated by fields relatively local to the currents. As a by-product one classifies the local extensions of \(\mathcal{A}\) by integer spin chiral fields. For each extended observable algebra (satisfying Haag duality [5]) one should be able to construct a 2-dimensional CFT whose state space contains all admissible superselection sectors.

A breakthrough in classifying rational CFT came about somewhat surprisingly from Cardy’s study [6] of the effect of boundary conditions on the operator content of 2-dimensional CFT. It led rather quickly to the ADE classification of modular invariant partition functions [7] for the \(su_2\) conformal current algebras,
(where $k$ stands for the Kac-Moody central charge or level) as well as for minimal conformal models. Modular ($\text{SL}_2(\mathbb{Z})$) invariance is related to the global reparametrization invariance of string theory [8], which can be made manifest in the path integral formulation of quantum field theory, and to the possibility of extending a CFT to a higher genus Riemann surface [9]. The main objective of the present paper is to prove that the ADE classification of modular invariant CFT can also serve to describe all local extensions of the chiral current algebra $A_k$.

In one direction the relation is obvious (and long known): the $D_{2\rho+2}$ series associated with the level $k = 4\rho$ current algebra corresponds to an extension on $A_{4\rho}$ by an $A_k$-primary field of $SU_2$ weight (= twice isospin) $\lambda = 4\rho$ and conformal dimension

$$\Delta_{4\rho} = \frac{\lambda(\lambda + 2)}{4(k + 2)} = \rho \quad (= 1, 2, \ldots) ; \quad (1.3)$$

the (non-diagonal) $E_6$ model corresponds to an extension of $A_{10}$ by a (primary) current of weight

$$\lambda = 6, \quad \Delta_6 = 1 \quad (\text{for } k = 10) ; \quad (E6)$$

finally the $E_8$ model involves 3 new local primary fields of weights

$$(\lambda, \Delta) = (10, 1), (18, 3), (28, 7) \quad \text{for } k = 28 . \quad (E8)$$

There are, however, many more integer spin $A_k$-primary fields. If $k + 2 = p_1^{n_1} \cdots p_r^{n_r}$ is the decomposition of $k + 2$ into primes, then the number of primary fields of integer dimension (including the unit operator) is $2^{\nu-1}$. All these fields being relatively local to the $SU_2$ currents, the question arises: when are they local with respect to themselves? We give necessary conditions for this by analyzing the 4-point function of a local primary field. They imply that the only local extensions $A_k$ are those given above (corresponding to even non-diagonal series in the ADE classification). Earlier studies of the relation between modular invariants and local extensions of the chiral algebra are only concerned with (primary) spin 1 [10] and simple currents [11].

The main tool in our study is the Knizhnik–Zamolodchikov (KZ) equation [12, 13, 14] for the 4-point function of an $A_k$-primary field and the corresponding fusion rules which fix the boundary conditions. The requirement that its solution (for integer $\Delta$) should be a rational Möbius (and $SU_2$) invariant function of the world sheet variables $z_i$ singles out the representations that mix (i.e., their characters appear in products) with the vacuum one in modular invariant partition functions.

The polynomial solutions of the KZ equation are unique (for given $k$ and $\lambda$) and therefore invariant under the 6-element ("crossing") symmetry group

$$S_4/(Z_2 \times Z_2) \simeq D_3 \simeq W(A_2) \quad (1.4)$$

that reflects local commutativity for Bose fields. Here $S_4$ is the permutation group of 4 objects, $Z_2^2$ is a 4-element normal Abelian subgroup of $S_4$, $D_3$ is the dihedral (triangle) group, $W(A_2)$ is the Weyl group of the root system of $SU_3$.

We exhibit in Sec. 3 a complete set of polynomial $D_3$-invariants of two variables. A systematic discussion of the finite symmetry of the solutions has, in our view, an interest of its own; it is applied in Sec. 5 for writing these solutions in a manifestly $D_3$-invariant form. It certainly goes beyond what is needed (and is already contained in the pioneer work on 2D CFT [15]) for deriving the main result of the present paper, the complete classification of rational solutions of the KZ equation in Sec. 4.

2. KZ EQUATION FOR THE 4-POINT FUNCTION OF AN $A_k$-PRIMARY FIELD. A SIMPLE UNIQUENESS PROPERTY

We shall use the polynomial realization of $SU_2$ tensors of [13, 14]. The $SU_2$ current

$$J(z, \zeta) = J_-(z) + 2\zeta J_3(z) - \zeta^2 J_+(z) \quad (2.1)$$
(where \( z \) is the world sheet coordinate and the formal variable \( \zeta \) is a substitute for an isospin index) satisfies the commutation relations

\[
[J(z_1, \zeta_1), J(z_2, \zeta_2)] = -\delta(z_{12})(\zeta_{12}^2 \partial_\zeta + 2\zeta_{12})J(z_2, \zeta_2) + k\zeta_{12}^2 \delta'(z_{12}),
\]

\( \zeta_{12} = \zeta_1 - \zeta_2 \); \( z_{12} = z_1 - z_2 \); the \( \delta \)-function on the circle is normalized by

\[
\int_{|z_2|=|z_1|} \delta(z_{12}) f(z_2) \frac{dz_2}{2\pi i} = f(z_1).
\]

An \( \mathcal{A}_k \) primary field \( V = V_\lambda \) of \( SU(2) \) weight \( \lambda \) and conformal dimension

\[
\Delta = \Delta_\lambda = \frac{1}{4} \frac{\lambda(\lambda + 2)}{k + 2}, \quad \lambda = 0, 1, 2, \ldots, k
\]

is viewed as a polynomial of degree \( \lambda \) in \( \zeta \) and is characterized by the Ward identities

\[
\langle 0 | J(z_1, \zeta_1) V(z_2, \zeta_2) = -\frac{1}{z_{12}} (\zeta_{12}^2 \partial_{\zeta_2} + \lambda\zeta_{12}) \langle 0 | V(z_2, \zeta_2),
\]

\[
V(z_1, \zeta_1) J(z_2, \zeta_2) \langle 0 = \frac{1}{z_{12}} (\zeta_{12}^2 \partial_{\zeta_1} - \lambda\zeta_{12}) V(z_1, \zeta_1) \langle 0
\]

and the KZ equation

\[
(k + 2) \partial_z V =: \frac{\lambda}{2} (\partial_\zeta J)V - J \partial_z V.
\]

Here the normal product is defined by the non-singular term in the operator product expansion; for instance

\[
: J_3(z_1) V(z_2, \zeta) := J_3(z_1) V(z_2, \zeta) - \frac{1}{z_{12}} \left( \zeta \partial_\zeta - \frac{\lambda}{2} \right) V(z_2, \zeta).
\]

As a consequence of (2.5) and (2.6) the 4-point function

\[
W_4 = \langle 0 | V(z_1, \zeta_1) \cdots V(z_4, \zeta_4) | 0 \rangle
\]

satisfies the KZ equation

\[
\left\{ (k + 2) \partial_{z_2} + \frac{\Omega_{12}}{z_{12}} - \frac{\Omega_{23}}{z_{23}} - \frac{\Omega_{24}}{z_{24}} \right\} W_4 = 0
\]

where \( \Omega_{ij} \) are \( SU_2 \) invariant operators acting in the tensor product space of polynomials of 2 variables \( \zeta_i \) and \( \zeta_j \), \( (i < j) \),

\[
\Omega_{ij} = 2I_i I_j = \frac{1}{2} \lambda^2 - \lambda \zeta_{ij} (\partial_i - \partial_j) - \zeta_{ij}^2 \partial_i \partial_j
\]

and satisfying (as a consequence of isospin conservation, \( I_1 + I_2 + I_3 + I_4 = 0 \))

\[
\Omega_{12} + \Omega_{23} + \Omega_{24} + \frac{1}{2} \lambda(\lambda + 2) = 0.
\]

Möbius and \( SU_2 \) invariance allow us to write \( W_4 \) in the form

\[
W_4(z_1 \zeta_1, \ldots, z_4 \zeta_4) = (z_{12} z_{23} z_{34} z_{14} z_{13} z_{24})^{-2\Delta} H(\vec{\xi}; \vec{\eta})
\]

where \( H \) is a homogeneous polynomial of degree \( \lambda \) in

\[
\vec{\xi} = (\xi_1, \xi_2), \quad \xi_1 = \zeta_{12} \zeta_{34}, \quad \xi_2 = \zeta_{14} \zeta_{23}
\]
and a homogeneous function of degree $4\Delta$ in

$$\vec{\eta} = (\eta_1, \eta_2), \quad \eta_1 = z_{12} z_{34}, \quad \eta_2 = z_{14} z_{23}$$

(2.12b)

Inserting (2.11) into (2.9), noting that

$$\frac{\eta_1 + \eta_2}{z_{24}} = \frac{\eta_2}{z_{23}} - \frac{\eta_1}{z_{12}} (= z_{13})$$

(2.12c)

and equating the coefficients to $z_{12}^{-1}$ and to $z_{23}^{-1}$ separately to 0 we find

$$(k + 2)\eta_1 \frac{\partial}{\partial \eta_1} H = \left\{ (1 - \eta)\Omega_{12} - \eta \Omega_{23} + \frac{1}{2} \lambda(\lambda + 2) \right\} H,$$

(2.13a)

$$(k + 2)\eta_2 \frac{\partial}{\partial \eta_2} H = \left\{ \eta \Omega_{23} - (1 - \eta)\Omega_{12} + \frac{1}{2} \lambda(\lambda + 2) \right\} H,$$

(2.13b)

where

$$\eta = \frac{\eta_1}{\eta_1 + \eta_2} = \frac{z_{12} z_{34}}{z_{13} z_{24}}.$$  

(2.14)

The sum of the two equations reproduces the Euler homogeneity condition for $\Delta$, $\lambda$ and $k$ related by (2.4); their difference can be written as

$$\left\{ (k + 2)\partial_{\eta} - \frac{\Omega_{12}}{\eta} + \frac{\Omega_{23}}{1 - \eta} \right\} H(\vec{\eta}; \eta, 1 - \eta) = 0.$$  

(2.15)

If we set

$$H(\vec{\xi}; \vec{\eta}) = (\xi_1 + \xi_2)^\lambda (\eta_1 + \eta_2)^{4\Delta} P(\xi, \eta),$$

(2.16)

where $\xi$ is the counterpart of $\eta$ (2.14),

$$\xi = \frac{\xi_1}{\xi_1 + \xi_2} = \frac{\xi_{12} \xi_{34}}{\xi_{13} \xi_{24}},$$

(2.17)

then Eq. (2.15) takes the form of a partial differential equation

$$(k + 2)\eta(1 - \eta) \frac{\partial P}{\partial \eta} = \left\{ \lambda[\lambda + 1 - (\lambda + 2)\eta - \lambda \xi] \\ - [\xi^2 - 2\xi \eta + \eta + 2\lambda \xi(1 - \xi)] \frac{\partial}{\partial \xi} + \xi(1 - \xi)(\xi - \eta) \frac{\partial^2}{\partial \xi^2} \right\} P(\xi, \eta).$$

(2.18)

Let now $\Delta$ be an integer; this implies, according to (2.4) that the $SU_2$ weight $\lambda$ is even

$$\lambda \geq 4\Delta \quad \lambda = 4, 6, 8, \ldots.$$  

(2.19)

The 4-point function (2.11) (2.16) is then a rational (single-valued) function of $\{z_i\}$ iff $P$ is a polynomial in $\eta$ of degree $4\Delta$ ($= 4, 8, \ldots$).

The standard normalization of the 2-point function of $V_\lambda$ and the factorization property of $W_4$ into two 2-point functions for small $z_{12}$ yield the “initial condition”

$$\lim_{\eta_1 \to 0} \{ \eta_1^{2\Delta} W_4 \} = H(\vec{\eta}; 0, 1) = \xi_1^\lambda$$

(2.20a)

or

$$P(\xi, 0) = \xi^\lambda.$$  

(2.20b)

Clearly, this simple requirement implies uniqueness of the analytic solution of (the first order in $\eta$) Eq. (2.18). Consistency then demands that Eq. (2.20) be equivalent to the “$u$-channel boundary condition”

$$\lim_{\eta_2 \to 0} \{ \eta_2^{2\Delta} W_4 \} = H(\vec{\xi}; 1, 0) = \xi_2^\lambda$$

(2.21a)

$$P(\xi, 1) = (1 - \xi)^\lambda.$$  

(2.21b)

This consistency condition is certainly verified for the solutions listed in Sec. 4 below. We have checked numerically that no other solutions exist for $\Delta \leq 10$. 

1066
3. LOCALITY AND BOSE STATISTICS: THE $D_3$-SYMMETRY

Locality and Bose statistics require the permutation ($S_4$) symmetry of the rational 4-point function. To exhibit the $S_4$ action on the homogeneous polynomial $H(\xi, \eta)$ introduced in (2.11) we recall that $S_4$ is generated by the three transpositions $s_i = (i i + 1)$, $i = 1, 2, 3$, subject to the relations

$$s_1 s_3 = s_3 s_1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad i = 1, 2, \quad (3.1a)$$
$$s_i^2 = 1, \quad i = 1, 2, 3. \quad (3.1b)$$

We first note that $S_4$ has a four-element invariant subgroup $Z_2 \times Z_2$, generated by two commuting involutive elements $s_1 s_3$ and $s_2$ (14) where $(14) = s_1 s_2 s_3 s_2 s_1$; it acts trivially on $\xi$ and $\eta$. This is an immediate consequence of the $2 \times 2$ matrix realization of $s_i$:

$$s_1 x_1 = -x_1, \quad s_1 x_2 = x_1 + x_2, \text{ or } s_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = s_3, (x_i = \xi_i \text{ or } \eta_i)$$
$$s_2 x_1 = x_1 + x_2, \quad s_2 x_2 = -x_2, \text{ or } s_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = s_1 s_3 s_2 s_1 = (14). \quad (3.2)$$

The 6-element factor group

$$D_3 = S_4 / (Z_2 \times Z_2) \quad (3.3)$$

is isomorphic to the Weyl reflection group for $A_2 \simeq su_3$. (To see this it suffices to identify $x_i$ with the simple roots of $A_2$.) $D_3$ is generated by any pair of reflections among $s_1, s_2$, and

$$s := s_1 s_2 s_1 = s_2 s_1 s_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = (= (13)),$$ \quad (3.4)

or, alternatively, by $s$ and the $Z_3$ generator

$$\omega := s_1 s_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = (\omega^3 = 1). \quad (3.5)$$

Setting $g : H(\xi, \eta)^2 H(g^{-1} \xi, g^{-1} \eta)$ and using (2.16) we derive the following $D_3$ action in the space of polynomials $P(\xi, \eta)$:

$$s_1 : P(\xi, \eta) \rightarrow (1-\xi)^{\lambda}(1-\eta)^{4\Delta}P\left(\frac{\xi}{\xi - 1}, \frac{\eta}{\eta - 1}\right), \quad (3.6a)$$
$$s_2 : P(\xi, \eta) \rightarrow \xi^{\lambda} \eta^{4\Delta}P\left(\frac{1}{\xi}, \frac{1}{\eta}\right), \quad (3.6b)$$
$$s : P(\xi, \eta) \rightarrow (-1)^{\lambda+4\Delta}P(1-\xi, 1-\eta), \quad (3.6c)$$
$$\omega : P(\xi, \eta) \rightarrow (-\xi)^{\lambda}(-\eta)^{4\Delta}P\left(\frac{\xi - 1}{\xi}, \frac{\eta - 1}{\eta}\right). \quad (3.6d)$$

The $D_3$ invariance of the KZ equation is easiest to verify for Eq. (2.13). To this end we need the relations

$$\Omega_{ij} = \Omega_{ji} \quad \Omega_{13} = \Omega_{24} \quad (3.7)$$

(on the top of (2.10)) and the contragradient transformation law for derivatives

$$\frac{\partial}{\partial \eta'_i} = (g^{-1})_{ij} \frac{\partial}{\partial \eta_j} \quad \text{for} \quad \eta'_j = \eta_j g_{ij}. \quad (3.8)$$
The uniqueness property noted in the previous section implies that if a polynomial solution of Eq. (2.13) or (2.18) exists then it is necessarily $D_3$ invariant.

We need to recall some results on the ring $\mathcal{P}^{D_3}$ of $D_3$-invariant polynomials $P(x)$, where $x = (\overrightarrow{x})$ is a vector of the 2-dimensional carrier space of the $D_3$ representation defined by (3.2) or (3.4) and (3.5). From Hilbert\(^1\) we know that $\mathcal{P}^{D_3}$ is finitely generated.

Moreover, since the representation of $D_3$ is generated by two reflections $s_1, s_2$, we know from a Chevalley’s theorem [17] that $\mathcal{P}^{D_3}$ is a two-variable polynomial ring, i.e., it is made of all two-variable polynomials $P(J_2, J_3)$, where $J_2(\overrightarrow{x}), J_3(\overrightarrow{x})$ are two homogeneous polynomials generating $\mathcal{P}^{D_3}$. Their respective degrees 2 and 3 are given by the Molien function

$$M(t) = \sum d_n t^n = \frac{1}{6} \sum_{g \in D_3} \det(1 - tg)^{-1} = \frac{1}{(1-t^2)(1-t^3)}$$

The generators $J_2, J_3$, expressed alternatively in terms of the homogeneous variables $x_1, x_2$ and the ratio $x = x_1(x_1 + x_2)^{-1}$, are

$$J_2(\overrightarrow{x}) = x_1^2 + x_2^2 + x_1x_2 \equiv \overrightarrow{x} \cdot \overrightarrow{x} = \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (3.10a)$$

$$J_2 = x^2 + (1-x)^2 + x(1-x) = 1 - x + x^2, \quad (3.10b)$$

$$J_3(\overrightarrow{x}) = (2x_1 + x_2)(x_1 + 2x_2)(x_1 - x_2), \quad (3.11a)$$

$$J_3 = (1 + x)(2 - x)(2x - 1). \quad (3.11b)$$

It is also known that the pseudo-invariants are of the form $\mathcal{P}^{D_3} Q_3(\overrightarrow{x})$, where $Q_3$ is the product of the left-hand sides of the 3 reflection plane equations, which is also equal to the Jacobian

$$Q_3(\overrightarrow{x}) = x(1-x). \quad (3.12a)$$

$$Q_3 = x(1-x). \quad (3.12b)$$

Its square is an invariant:

$$27Q_3^2 = 4J_2^3 - J_3^2. \quad (3.13)$$

The tensor square of the 2 dimensional irreducible representation decomposes into

$$2 \otimes 2 = 1^+ \oplus 1^- \oplus 2 \quad (3.14)$$

where $1^+, 1^-$ are respectively the trivial and the pseudo-invariant representations (only the latter appears in the antisymmetric part of the tensor product: it corresponds to the determinant of two vectors). The appearance of the representation 2 in the symmetric part of the tensor product $2 \otimes 2$ implies the existence of a symmetric nonassociative algebra (see [18]) with $D_3$ as a group of automorphisms. We denote its composition law by $\vee$ and compute it explicitly. By definition

$$(\overrightarrow{x} \vee \overrightarrow{y}) \equiv j_3(\overrightarrow{x} \vee \overrightarrow{y}, \overrightarrow{z}) = j_3(\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}) \quad (3.15a)$$

where $j_2, j_3$ are the completely symmetric multilinear forms obtained by polarization of the invariants $J_2, J_3$:

$$j_2(\overrightarrow{x}, \overrightarrow{y}) \equiv \overrightarrow{x} \cdot \overrightarrow{y} \equiv \frac{1}{2}(J_2(\overrightarrow{x} + \overrightarrow{y}) - J_2(\overrightarrow{x}) - J_2(\overrightarrow{y})) \quad (3.15b)$$

$$= \frac{1}{2}(2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2);$$

\(^1\)For a review of concepts and results needed below and for references to the original sources – see [16].
\[ j_3(\vec{x}, \vec{y}, \vec{z}) = 2x_1y_1z_1 + x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1 
- x_1y_2z_2 - x_2y_1z_2 - x_2y_2z_1 - 2x_2y_2z_2 ; \]

they satisfy: \( j_2(\vec{x}, \vec{z}) = J_2(\vec{x}), j_3(\vec{x}, \vec{z}) = J_3(\vec{x}) \); as a result,

\[ \vec{x} \vee \vec{y} = 2 \left( x_1y_1 + x_1y_2 + x_2y_1 \right) \]

The second Chevalley's theorem in [17] says that the ring of polynomials \( \mathcal{P}(\vec{x}) \) is a free \( \mathcal{P}_{D3} \)-module corresponding to the regular representation \( \text{Reg} = 1 \oplus 2 \oplus 2 \oplus 2 \) of \( D_3 \); its basis is given by the polynomials of respective degrees 0, 1, 1, 2, 2, 3:

\[ 1, x_1, x_2, (\vec{x} \vee \vec{y})_1, (\vec{x} \vee \vec{y})_2, Q_3(\vec{x}) . \]

We denote by \( P_{mn}(\vec{\xi}, \vec{\eta}) \) the polynomials homogeneous in \( \xi_i \) of degree \( m \) and in \( \eta \) of degree \( n \). They generate a ring which is the tensor product \( \mathcal{P}(\vec{\xi}) \otimes \mathcal{P}(\vec{\eta}) \); this is therefore a (36-dimensional) free module over the polynomial ring \( \mathcal{P}(J_2(\vec{\xi}), J_3(\vec{\xi})) \otimes \mathcal{P}(J_2(\vec{\eta}), J_3(\vec{\eta})) = \mathcal{P}(J_{20}, J_{30}, J_{02}, J_{03}) \), where we have used the shorter expressions \( J_{20} \) for \( J_2(\vec{\xi}) \), \( J_{30} \) for \( J_3(\vec{\xi}) \), \( J_{02} \) for \( J_2(\vec{\eta}) \), \( J_{03} \) for \( J_3(\vec{\eta}) \). Since \( \text{Reg} \otimes \text{Reg} = 6\text{Reg} \), the ring of \( D_3 \)-invariant polynomials \( (\mathcal{P}(\vec{\xi}) \otimes \mathcal{P}(\vec{\eta}))(\mathcal{P}_{D3} \) is a (6-dimensional) free module over the polynomial ring \( \mathcal{P}(J_{20}, J_{30}, J_{02}, J_{03}) \) with basis:

\[ J_{00} = 1 \quad J_{11} = 2\vec{\xi} \cdot \vec{\eta}, \quad J_{21} = (\vec{\xi} \vee \vec{\xi}) \cdot \vec{\eta}, \quad J_{12} = \vec{\xi} \cdot (\vec{\eta} \vee \vec{\eta}), \]

\[ J_{22} = \frac{1}{2}(\vec{\xi} \vee \vec{\xi}) \cdot (\vec{\eta} \vee \vec{\eta}), \quad J_{33} = Q_{30}(\vec{\xi}) Q_{03}(\vec{\eta}) . \]

Similarly, the pseudo-invariants of \( \mathcal{P}(\vec{\xi}) \otimes \mathcal{P}(\vec{\eta}) \) form a (6-dimensional) free module over the polynomial ring \( \mathcal{P}(J_{20}, J_{30}, J_{02}, J_{03}) \) with basis:

\[ Q_{30} = Q_3(\vec{\xi}), \quad Q_{03} = Q_3(\vec{\eta}), \quad Q_{11} = \det(\vec{\xi}, \vec{\eta}), \quad Q_{21} = \frac{1}{2} \det(\vec{\xi} \vee \vec{\xi}, \vec{\eta}) = 3q_3(\vec{\xi}, \vec{\xi}, \vec{\eta}), \]

\[ Q_{12} = \frac{1}{2} \det(\vec{\xi}, \vec{\eta} \vee \vec{\eta}) = 3q_3(\vec{\xi}, \vec{\eta}, \vec{\eta}), \quad Q_{22} = \det(\vec{\xi} \vee \vec{\xi}, \vec{\eta} \vee \vec{\eta}), \]

where \( \det(\vec{x}, \vec{y}) = x_1y_2 - x_2y_1 \) and \( q_3 \) is the trilinear form obtained by polarization of the pseudo-invariant:

\[ q_3(\vec{x}, \vec{z}) = Q_3(\vec{x}) . \]

Of course this module of pseudo-invariants is not a ring while that of the invariants is. We can substitute \( J_{22} \) by \( Q_{11}^2 \) as one of the 6 basis elements (3.17) of the ring of invariants since

\[ J_{22} = 2J_{20}J_{02} - 3Q_{11}^2 . \]

The mixed invariants \( J_{\alpha, \beta} \) can be written alternatively in terms of the homogeneous coordinates \( \vec{\xi}, \vec{\eta} \) and of the ratios (2.14), (2.17):

\[ J_{11}(\vec{\xi}, \vec{\eta}) = 2\xi_1\eta_1 + 2\xi_2\eta_2 + \xi_1\eta_2 + \xi_2\eta_1, \quad J_{11} = J_{11}(\xi, 1 - \xi; \eta, 1 - \eta) = 2(1 + \xi\eta) - \xi - \eta. \]

\[ J_{21}(\vec{\xi}, \vec{\eta}) = 3(\xi_1^2\eta_1 - \xi_2^2\eta_2) - (\xi_1 - \xi_2)^2(\eta_1 - \eta_2), \]

\[ J_{21} = (1 - 2\eta)(\xi^2 - 4\xi + 1) - 3(1 - 2\xi)(1 - \eta) . \]
The expressions for the invariant solutions of the KZ equations (Sec. 5 below) simplify if we also use as invariants even products of pseudoinvariants such as

\[ Q_{21}(\xi, \eta) = \xi_1 \eta_2 (\xi_1 + 2\xi_2) + \xi_2 \eta_1 (2\xi_1 + \xi_2), \]  
\[ Q_{21} = (1 - 2\xi)\eta + \xi (2 - \xi). \]  

(3.22a)  

(3.22b)

We shall use higher powers of \( Q_{11}^2 \), which could be excluded, using the relations:

\[ 3Q_{11}^2 = 4J_{20}J_{02} - J_{11}^2. \]  
\[ Q_{11} = Q_{11}^2 J_{20}J_{02} + 3J_{11}Q_{30}Q_{03} - Q_{30}Q_{12}J_{02} - Q_{03}Q_{21}J_{20}. \]  

(3.23)  

(3.24)

4. CLASSIFICATION OF ALL RATIONAL 4-POINT FUNCTIONS

The unique polynomial solution of Eq. (2.18) is consistent with a truncated Gepner-Witten [19] fusion rule

\[ \lambda \times \lambda = \sum_{I=0}^{\min(\lambda, k-\lambda)} N_{\lambda\lambda 2I} 2I \]  

(4.1)

where the multiplicity \( N_{\lambda\lambda 2I} \) is positive if the primary field of weight \( 2I \) is local so that, in particular, it has an integer dimension

\[ \Delta_{2I} = \frac{I(I+1)}{k+2} \in \{0, 1, \ldots, \Delta = \Delta_\lambda\}, \quad I \leq \lambda, \quad I \leq k - \lambda. \]  

(4.2)

In order to take into account the contribution of such intermediate states to the 4-point function we shall expand \( P \) in the “regular basis” \( \xi^{\lambda-l}(1-\xi)^l \) of [14, 20]:

\[ P(\xi, \eta) = \sum_{l=0}^{\lambda} \binom{\lambda}{l} \xi^{\lambda-l}(1-\xi)^l f_l(\eta) \]  

(4.3)

where each \( f_l \) is a polynomial of degree not exceeding \( 4\Delta \).

Inserting (4.3) into (2.18) we find the following system of ordinary differential equations for \( f_l \):

\[ (k + 2)\eta(1 - \eta) \frac{d}{d\eta} f_l(\eta) = \{ l(l + 1)(1 - \eta) - (\lambda - l) \} \]
\[ \times (\lambda - l + 1)\eta f_l(\eta) + (l + 1)(\lambda - l)(1 - \eta)f_{l+1}(\eta) \]
\[ - l(\lambda + 1 - l)\eta f_{l-1}(\eta), \quad l = 0, 1, \ldots, \lambda. \]  

(4.4)

The symmetry of \( P \) under the duality transformation (3.6c) implies the relation

\[ f_l(\eta) = f_{\lambda-l}(1 - \eta). \]  

(4.5)

The integral representation of a general (not necessarily polynomial) solution of the KZ equation (see Eqs. (24)-(27) of [14]) implies that the small \( \eta \) behavior of the \( s \)-channel contribution of an isospin \( I \) primary field to the “partial wave” \( f_\lambda \) is

\[ f_{\lambda I}(\eta) \sim \eta^{\lambda-I+\Delta_{2I}} \quad \text{for} \quad \eta \to 0 \quad \left( I - \Delta_{2I} = I \frac{k+1-I}{k+2} \geq 0 \right). \]  

(4.6)

(In deriving (4.6) one uses the triangular transition matrix which relates the \( s \)-channel basis to the regular one.)
Proposition 1. Every polynomial solution of the system (4.4) obeying the symmetry relation (4.5) and the initial condition (2.20) satisfies

\[ f_\lambda(\eta) = \eta^{4\Delta}, \quad f_0(\eta) = (1 - \eta)^{4\Delta}. \] (4.7)

Proof. Owing to (4.6) Proposition 1 follows from the inequality

\[ \lambda - I + \frac{I(I + 1)}{k + 2} \geq \frac{\lambda(\lambda + 2)}{k + 2} = 4\Delta, \] (4.8)

since the degree of the polynomial \( f_\lambda(\eta) \) does not exceed \( 4\Delta \). To establish (4.8) it suffices to prove the weaker inequality

\[ \lambda - I + \frac{I(I + 1)}{k + 2} \geq \frac{\lambda(\lambda + 1)}{k + 2}, \] (4.9)

noting that \( 4\Delta \) is the smallest integer that exceeds the right-hand side, since

\[ 0 < 4\Delta - \frac{I(I + 1)}{k + 2} = \frac{\lambda}{k + 2} < 1. \]

The inequality (4.9) is equivalent to

\[ I(k + 1 - \lambda) \leq \lambda(k + 1 - \lambda) \quad \text{for} \quad I \leq \min(\lambda, k - \lambda). \] (4.10)

The quadratic function \( f(x) = x(k + 1 - x) \) has a maximum for \( x = \frac{1}{2}(k + 1) \) and is symmetric around that point. Hence, for \( \lambda \leq \frac{1}{2}(k + 1) \), (4.10) follows because \( I \leq \lambda \); for \( \lambda > \frac{1}{2}(k + 1) \) it is true, since \( I < k + 1 - \lambda \). The normalization of \( f_\lambda \) follows from the symmetry relation \( f_0(\eta) = f_\lambda(1 - \eta) \) implied by (4.5) and the initial condition (2.20).

Proposition 2. If Eqs. (4.4) admit a polynomial solution satisfying (4.7), then either

\[ \lambda = k = 4\Delta \quad (D_{2\Delta+2} \text{ series}) \] (4.11)

\[ f_1(\eta) = \eta^4(\eta - 1)^{\lambda - I}, \quad l = 0, 1, \ldots, \lambda, \] (4.12)

or there exists a local primary field \( V_{2I} \) of dimension

\[ \Delta_{2I} = 1 \Leftrightarrow k + 2 = I(I + 1), \quad I \geq 2 \] (4.13)

that appears in the operator product expansion

\[ z_1^{2\Delta} V_\lambda(z_1, \zeta_1) V_\lambda(z_2, \zeta_2) = \zeta_{12}^\lambda - \frac{\lambda}{k} \zeta_{12}^{\lambda - I} \left( 1 + \frac{1}{2} \zeta_{12} \partial_2 \right) \] (4.14)

\[ \times \int_{z_2}^{z_1} J(z, \zeta_2) \, dz + C_{\lambda 2I} \zeta_{12}^{\lambda - I} \sum_{n=0}^{I} \left( \frac{I}{n} \right) \frac{(2I - n)!}{(2I)!} \]

\[ \times (\zeta_{12} \partial_2)^n \int_{z_2}^{z_1} V_{2I}(z, \zeta_2) \, dz + O(z_{12}^2) \]

with a non-zero structure constant \( C_{\lambda 2I} \).

Proof. Equation (4.4) for \( l = 0 \) together with (4.7) gives

\[ f_1(\eta) = -\eta(1 - \eta)^{4\Delta - 1} = f_{\lambda - 1}(1 - \eta). \] (4.15)
If a partial wave \( f_l \) behaves like \( cr \eta \) for small \( \eta \) and \( l \geq 2 \), then the second alternative of Proposition 2 (Eq. (4.13)) should take place as a consequence of (4.14). If such a term is absent, then the small \( \eta \) behavior of \( P \) would read

\[
P(\xi, \eta) = \xi^\lambda + [(\lambda - 4\Delta)\xi - \lambda] \xi^{\lambda-1} \eta + O(\eta^2).
\]

Applying now the symmetry property (3.6a) (for even \( \lambda \) and \( 4\Delta \)) and keeping just the constant and linear term in \( \eta \) we find

\[
P(\xi, \eta) = \xi^\lambda - \lambda \xi^{\lambda-1} \eta + O(\eta^2).
\]

This is only compatible with (4.16) for \( \lambda = 4\Delta (\pm k) \).

The \( D_{2\Delta+2} \) solution is, in fact, given by

\[
P(\xi, \eta) = (\xi - \eta)^k, \quad k = 4\Delta.
\]

(It also makes sense for odd \( 2\Delta \), corresponding in that case to the 4-point function of a Fermi field.)

It thus remains to list the models involving a \( \Delta = 1 \) local primary current with \( \lambda > 4 \) (the case \( \lambda = 4 \) corresponding to the \( D_4 \) model).

**Proposition 3.** The system (4.4) has a solution for \( \Delta = 1, \lambda > 4 \) in just two cases:

\[
k = 10, \quad \lambda = 6 \quad (E_6) \tag{4.19a}
\]

\[
k = 28, \quad \lambda = 10 \quad (E_8) \tag{4.19b}
\]

**Proof.** From Proposition 1 and Eq. (4.15) we have

\[
f_\lambda = \eta^4, \quad f_{\lambda-1} = -\eta^3(1 - \eta).
\]

After three more iterations we obtain

\[
f_{\lambda-4}(0) = \frac{(\lambda - 6)(\lambda - 10)\lambda(3\lambda - 4)(\lambda^2 - 8\lambda + 4)}{768(\lambda - 1)(\lambda - 2)(\lambda - 3)}.
\]

On the other hand, it follows from (2.20) that

\[
f_l(0) = \delta_{l0}.
\]

This is only consistent with (4.20) (and \( \lambda > 4 \)) for \( \lambda = 6 \) or \( \lambda = 10 \).

In the case (4.19a) there are three Fermi and Bose local fields with weights

\[
(\lambda, \Delta) = (4, \frac{1}{2}), (6, 1), (10, \frac{5}{2}) \quad \text{for} \quad k = 10.
\]

For \( k = 28 \) one has just the three \( E_8 \) Bose fields of weights listed in Sec. 1. We shall exhibit the polynomial solutions of Eq. (2.18) for all these cases in Sec. 5 below. We have thus established the main results of this paper.

**Theorem.** All polynomial solutions of Eq. (2.18) satisfying the equivalent boundary conditions (2.20), (2.21) correspond either to integer \( \Delta \), and then they are given by the \( D_{\text{even}} (= D_{2\Delta+2}) \) series and by the \( (\lambda, \Delta) = (6, 1) \) \( E_6 \)-current or the \( (\lambda, \Delta) = (10, 1), (18, 3) \) fields of the \( E_8 \) model; or they are given by the half integer dimension fields with \( 2\Delta = \frac{1}{2} k \in \{1, 3, 5, \ldots \} \) (related to the \( D_{\text{odd}} \) series) or the quintet of \( \Delta = \frac{1}{2} \) isospin 2 fields of the \( E_6 \) mode.

### 5. THE EXCEPTIONAL SOLUTIONS AS POLYNOMIALS IN THE BASIC INVARIANTS

The 4-point function of the spin \( \frac{1}{2} \) Fermi field of isospin 2 for \( k = 10 \) is given by (2.11), (2.16) with

\[
P_{42} = P(\xi, \eta; \lambda = 4, 4\Delta = 2) = \xi^4 - 4\eta\xi^3 + (6\xi^2 - 4\xi)\eta + \eta^2.
\]
Its $D - 3$ symmetry becomes manifest if we write it in the form

$$P_{42} = J_{20}Q_{11}^2 + Q_{30}Q_{12}.$$  \hspace{1cm} (5.2)

The $\Delta = 1$, $\lambda = 6$ field of the $E_6$ model has a 4-point function proportional to (5.2):

$$P_{64} = \{(\xi^2 - \eta)^2 - 4\xi\eta(1 - \xi)^2\}(\xi - \eta)^2 = P_{42}Q_{11}^2.$$  \hspace{1cm} (5.3)

The same type of relation appears between the two exceptional polynomial solutions associated with the $E_6$ ($k = 28$) model. The 4-point function for the ($\eta = 10$, $\Delta = 1$) field

$$P_{10} = (\xi, \eta) = \left[\frac{(\xi + 1)(2\xi - 1)(\xi - 2)(\xi - 2)}{(1 - \xi + (\xi - 2)(2\xi - 1)) (2\xi - 1) \eta + \xi(\xi - 2)}\right]$$

(each of the two terms in the sum being $D_3$ invariant separately) is written as a combinations of 7 basic invariants:

$$P_{10} = J_{20}J_{02}Q_{11}^2 + 6J_{11}Q_{30}Q_{03}(J_{20}^3 + 4Q_{30}^2)$$

$$+ 3Q_{30}^2J_{20}J_{02}(J_{20}J_{02} - Q_{11}^2) + J_{20}Q_{21}Q_{03}(J_{20}^3 + 18Q_{30}^2)$$

$$+ Q_{30}Q_{12}J_{02}(Q_{30}^2 + 4J_{20}^2)$$

where $Q_{11}^2$ is expressed in terms of $J_{22}$ and $J_{20}J_{02}$ according to (3.19). The polynomial $P$ corresponding to the $\Delta = 3$ field is then

$$P_{18} = P_{10} Q_{11}^2.$$  \hspace{1cm} (5.5)

The fact that the invariant $J_{11}$ and the pseudoinvariants $Q_{12}$ and $Q_{21}$ only appear linearly in the above expressions is not an accident. It is a consequence of (3.23) and the identities

$$Q_{21}^2 = J_{20}Q_{11}^2 - 3Q_{30}Q_{12}, \hspace{0.5cm} Q_{12}^2 = J_{02}Q_{11}^2 - 3Q_{21}Q_{03},$$

$$Q_{21}Q_{12} = J_{11}Q_{11}^2 + 9Q_{30}Q_{03}.$$  \hspace{1cm} (5.6a)

The substitution of even powers of $J_{30}$ by polynomials of $Q_{30}^2$ and $J_{20}^3$ according to (3.13) has the advantage that the invariant solution of $P_{10}$ of the KZ equations involves only integer coefficients in front of the product of elementary invariants. This property would not hold if we exchange $Q_{30}Q_{12}$ by

$$9Q_{30}Q_{12} = 2J_{20}(2J_{20}J_{02} - 3Q_{11}^2) - J_{30}J_{12}$$

and use a similar expression for $Q_{21}Q_{03}$.

We note that the prefactor in front of the homogeneous polynomial $H$ in the expression (2.11) for $W_4$ is a negative power of a basic pseudoinvariant, $Q_3^2(\bar{\eta})^{-2\Delta}$. It guarantees that for odd $2\Delta$ $W_4$ is a pseudoinvariant, as it should be for the 4-point function of a local Fermi field.

We conclude with a few remarks concerning related open problems.

It would be interesting to extend the results of the present paper to higher rank current algebras for which no complete classification of modular invariant partition functions is available. An understanding is still missing of the appearance of polynomial solutions for special values of $\lambda$ and $\Delta$ from the existing integral representation [14] of the general solution of the KZ equation. Such an understanding could help extrapolate our results to minimal conformal models where a similar "Coulomb gas" representation for the 4-point function is available. Another way of approaching the classification of local extensions of the Virasoro algebra for minimal conformal models consists in exploring the differential equations coming from
null vector conditions. Recent progress in understanding these equations [21] does not seem, however, sufficient for our purposes. It should be noted that our results shed new light on the "naturality theorem" for maximally extended chiral algebras [22], which in turn indicates that they should indeed be valid well beyond the $su_2$ current algebra considered here.

We should also note that the polynomials $P$ normalized by (2.20) only have integer coefficients in front of $\xi^m\eta^n$, a fact that may have a deeper meaning.

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