ISOSPIN CONSTRAINTS BETWEEN THREE CROSS SECTIONS
AND TWO POLARIZATION DENSITY MATRICES

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We give the relation which isospin conservation imposes between the three cross sections and two polarization density matrices, for three reactions related by two isospin channels. It is valid for arbitrary spin and for spin correlations. Its application to the most usual cases of spin 1 and \( \frac{1}{2} \) is given.

Consider three reactions of the type

\[ a_1 + 2a_2 \rightarrow \gamma_3 + 4a_3 + \ldots \quad (\alpha = 1, 2, 3) \quad (1) \]

which have their corresponding particles (or resonances) in the same isospin multiplet, and which go through two isospin channels. For given energy momenta of the particles, each reaction is described by a transition matrix \( T_\alpha \) from the initial to the final polarization spaces. Then, isospin conservation imposes a linear relation between the three \( T_\alpha \)'s

\[ \sum_{\alpha=1}^{3} \gamma_\alpha T_\alpha = 0, \quad (2) \]

where the real numbers \( \gamma_\alpha \) are simple combinations of Clebsch-Gordan coefficients.

Let us denote by \( o_\alpha \) the differential cross sections and by \( s_\alpha \) the weighted cross sections

\[ s_\alpha = \sqrt{o_\alpha}. \quad (3) \]

It is well known that eq. (2) imposes a "triangular relation" between the square roots of the weighted cross sections, which can be written

\[ |s_3 - s_1 - s_2| (2s_1 s_2)^{-1} \leq 1. \quad (4) \]

In this letter we give the best relation imposed by isospin conservation on the three weighted cross sections \( s_1, s_2, s_3 \) and the two polarization density matrices \( \rho_1 \) and \( \rho_2 \) of the final state†. This relation is

\[ |s_3 - s_1 - s_2| (2s_1 s_2)^{-1} \leq \text{tr} \left( \sqrt{\rho_1} \rho_2 \sqrt{\rho_1} \right). \quad (5) \]

Note that for positive, trace one matrices \( \rho_1 \) and \( \rho_2 \), one has

\[ \text{tr} \left( \sqrt{\rho_1} \rho_2 \sqrt{\rho_1} \right) = \text{tr} \left( \sqrt{\rho_2} \rho_1 \sqrt{\rho_2} \right) \leq 1. \quad (6) \]

Eq. (5) can be applied in the following physical situations. The initial state is unpolarized or is equally polarized in the three reactions. The measured density matrices \( \rho_\alpha \) may be:

i) The joint density matrix of all final particles;

ii) The joint density matrix of some of them;

iii) The density matrix of a single final particle, say \( 3_\alpha \);

iv) The even multipole part of this matrix (i.e., that part which is most usually measured when \( 3_\alpha \) is a strongly decaying resonance).

When two cross sections and two polarizations are measured, eq. (5) yields the best bounds on the third cross section. When the three cross sections and one polarization are measured, eq. (5) defines the allowed domain for a second polarization.

First we sketch the proof of eq. (5). Then we give the explicit expression of its right hand side for the usually measured density matrices of spin 1 or \( \frac{3}{2} \) resonances‡.

We denote by \( \rho_0 \) the polarization density matrix of the initial state. It is convenient to introduce the weighted transition and density matrices

† There are no relations, besides eq. (4), between three cross sections and only one polarization density matrix.

‡ We do not consider here the case of spin \( \frac{1}{2} \) particles since we have already treated it completely in refs. [1, 2].
\[ M_\alpha = \gamma_\alpha T_\alpha \sqrt{\rho_\alpha}, \quad R_\alpha = s_\alpha \rho_\alpha, \]  
\[ R_\alpha = M_\alpha M_\alpha^*, \quad s_\alpha = \text{tr} R_\alpha. \]

Then eq. (2) reads
\[ \sum_{\alpha=1}^{3} M_\alpha = 0. \]

This equation can be written \(-M_3 = M_1 + M_2\), multiplication on the right by the adjoint equation yields
\[ R_3 - R_1 - R_2 = M_1 M_2^* + M_2 M_1^* \equiv 2 \text{Re} (M_2 M_1^*). \]

The trace of this expression and the use of Cauchy-Schwarz inequality yield eq. (4), but eq. (5) cannot be obtained by means of Cauchy-Schwarz inequality. It is derived from the polar decomposition [e.g., 4] of the transition operator \( M \):
\[ M_\alpha = \sqrt{M_\alpha M_\alpha^*}, \quad U_\alpha = \sqrt{R_\alpha U_\alpha}, \]
where \( R_\alpha \) is a positive operator acting on the final polarization space and \( U_\alpha \) is a partially isometric operator from the initial to the final polarization spaces. Then eq. (10) reads
\[ R_3 - R_1 - R_2 = 2 \text{Re} \sqrt{R_2} U_{21} \sqrt{R_1}. \]

where the unknown operator \( U_{21} = U_2 U_1^* \) can be any operator of norm \( \leq 1 \). Taking the absolute value of the trace of eq. (12) one obtains
\[ \sqrt{s_3 - s_1 - s_2} = \sqrt{\text{tr} \rho_1 \rho_2 U_{21}}. \]

To compute the maximum of the right hand side for any possible \( U_{21} \) we use now the polar decomposition for the operator \( \sqrt{\rho_1 \rho_2} \), i.e.,
\[ \sqrt{\rho_1 \rho_2} = \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \]
and we get
\[ \text{tr} \sqrt{\rho_1 \rho_2 U_{21}} \leq \text{Max} \text{Re} \sqrt{\rho_1 \rho_2 U_{12} U_{21}} \]
\[ = \text{tr} \sqrt{\rho_1 \rho_2 \rho_1}. \]

Indeed the maximum is reached for \( U_{21} = V_{12}^* \); therefore eq. (5) gives the best relation.

The proof given above is valid for the case i). To extend it for the cases ii) and iii) one may consider the non observed final particles as unpolarized initial particles. Joseph [5] proved the validity of eq. (5) for the case iv).

For \( 2 \times 2 \) matrices the right hand side of eq. (5) reduces to
\[ \text{tr} \sqrt{\rho_1 \rho_2 \rho_1} = \text{tr} \rho_1 + 2 \text{det} \rho_1. \]

This expression allows one to treat completely the usual cases of spin 1 and \( \frac{3}{2} \) particles produced in “\( B - \) symmetric reactions”, i.e., in parity conserving quasi two body reactions with unpolarized beam and target. Indeed for these cases, the density matrix, in transversity quantization, can be written as a direct sum of matrices of dimension 1 or 2. For simplicity we restrict ourselves to the usual situation where only the even part of the density matrix is measured (case iv).

a) Spin 1, B-symmetric, even polarization. We use the three orthonormal polarization parameters \( X, Y, Z \) which are related to the density matrix elements in transversity (T) and in helicity (H) quantization by
\[ Z = 3 T \rho_{11} - 1 = \frac{1}{4} - \frac{3}{4} (H \rho_{11} + \text{Re} H \rho_{1-1}) \]
\[ X = \sqrt{3} \text{Re} T \rho_{1-1} = \frac{1}{2} \sqrt{3} (-3 H \rho_{11} + \text{Re} H \rho_{1-1} + 1) \]
\[ Y = -\sqrt{3} \text{Im} T \rho_{1-1} = -\sqrt{6} \text{Re} H \rho_{10}. \]

The parameters must satisfy the positivity conditions
\[ -1 \leq Z \leq \frac{1}{4}, \]
\[ C(X, Y, Z) = (Z+1)^2 - 3(X^2 + Y^2) \geq 0, \]
which are the equations of a truncated cone [6]. In terms of these parameters for the matrices \( \rho_1 \) and \( \rho_2 \), eq. (5) can be written

\[ \text{tr} \rho_1 \rho_2 \rho_1 \rho_2 \rho_1 = \text{Max} \text{Re} \sqrt{\rho_1 \rho_2 \rho_1 \rho_2 \rho_1} \]

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This was felt by Kamei and Sasaki [3]; they have applied Cauchy-Schwarz inequality to each element of the matrix equation (10). Taking afterwards the trace, they obtained the relation
\[ \sqrt{s_3 - s_1 - s_2} = \frac{2 \text{Re} \sqrt{s_1 \rho_2}}{\sqrt{s_1 \rho_2}} \leq \text{Max} \text{Re} \sqrt{s_1 \rho_2} \]
and they remarked "there might exist a covariant expression of [this relation]; however we could not find it".
\[ 3|s_3 - s_1 - s_2|[(2\sqrt{s_1 s_2})^{-1} \leq \sqrt{1 - 2Z_1}\sqrt{1 - 2Z_2} + \]
\[ + \sqrt{2}(1+Z_1)(1+Z_2)+3(X_1 X_2 + Y_1 Y_2) + \]
\[ + \sqrt{C(X_1, Y_1, Z_1)C(X_2, Y_2, Z_2)}^{1/2}. \]

b) Spin \( \frac{1}{2} \), B-symmetric, even polarization. The orthonormal parameters \( X, Y, Z \) that we use now are related to the density matrix elements by
\[ Z = \frac{3}{4}\sqrt{3}\left(T_{\rho_{33} - \frac{1}{2}}\right) = -\frac{3}{4}\sqrt{3}\left(H_{\rho_{33} - \frac{1}{2}}\right) - 2\text{Re} H_{\rho_{33} - 1} \]
(20a)
\[ X = \frac{3}{4}\sqrt{3}\text{Re} T_{\rho_{33} - \frac{1}{2}} = -2\left(H_{\rho_{33} - \frac{1}{2}}\right) + \frac{3}{4}\sqrt{3}\text{Re} H_{\rho_{33} - 1} \]
(20b)
\[ Y = -\frac{3}{4}\sqrt{3}\text{Im} T_{\rho_{33} - \frac{1}{2}} = -\frac{3}{4}\sqrt{3}\text{Re} H_{\rho_{33} - 1}. \]
(20c)

They satisfy the positivity condition
\[ S(X, Y, Z) = 1 - 3(X^2 + Y^2 + Z^2) \geq 0, \]
(21)
which is the equation of a sphere [6]. Eq. (5) becomes in this case
\[ |s_3 - s_1 - s_2|[(2\sqrt{s_1 s_2})^{-1} \leq \]
\[ \sqrt{\frac{3}{2}}\left(1 + 3(X_1 X_2 + Y_1 Y_2 + Z_1 Z_2) + \]
\[ + \sqrt{S(X_1, Y_1, Z_1)S(X_2, Y_2, Z_2)}\right)^{1/2}. \]
(22)

If we look at eqs. (19) and (22) as constraints on \( \rho_2 \) (i.e., on \( X_2, Y_2, Z_2 \) when \( s_1, s_2, s_3 \) and \( \rho_1 \) are known, the allowed domain for \( \rho_2 \) is a pearl shaped convex domain which grows around the point \( \rho_1 \) when \( |s_3 - s_1 - s_2|[(2\sqrt{s_1 s_2})^{-1} \) decreases, and is contained in the truncated positivity cone for case a) or in the positivity sphere for case b). A more detailed study of this pearl shaped domain and of the relation between three cross sections and three polarizations will be carried on in a forthcoming publication.

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