SOME REMARKS ON POLARIZATION MEASUREMENT AND POLARIZATION DOMAIN

M. DAUMENS, G. MASSAS and P. MINNAERT
Laboratoire de Physique Théorique, Université de Bordeaux I †,
33170 Gradignan, France

and

L. MICHEL
Institut des Hautes Etudes Scientifiques,
91440 Bures Sur Yvette, France

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Abstract: We emphasize that any analysis of polarization measurement must be done in terms of the polarization domain which is much smaller than the domain of physical bounds. We propose a possible quantitative procedure to estimate the precision of a measurement and its compatibility with the positivity condition. We illustrate our discussion with the case of spin-$\frac{3}{2}$ particles and as an application we study some experimental results on $Y^*(1385)$. 

1. Introduction

In this paper we make some comments on the measurement of particle polarization in high-energy physics. In collaboration with M.G. Doncel, two of us (L.M. and P.M.) have lectured and written detailed notes on this subject [1, 2]. However, we feel necessary to refer again to this problem and emphasize some important points.

To illustrate our discussion we consider the non-trivial case of a spin-$\frac{3}{2}$ resonance, produced in a B-symmetric reaction [1, 2] (i.e. a parity-conserving reaction with unpolarized beam and target which is either a quasi-two-body reaction or a reaction analysed inclusively). As an application we study the results of an interesting experimental paper [3] which appeared recently. This paper, which will be referred to as AW, deals with the polarization of a $Y^*(1385)$ produced in the reactions:

\[ \pi^+ p \rightarrow K^+ Y^{**}, \quad K^0 p \rightarrow \pi^- Y^{**}, \tag{1} \]
\[ \pi^+ p \rightarrow X^+ Y^{**}, \quad K^0 p \rightarrow X^- Y^{**}. \tag{2} \]

† Equipe de recherche associée au CNRS.
This $Y^\ast(1385)$ is specially interesting, because from an observation of the cascade decays $Y^\ast \to \Lambda^0\pi^\ast$ (strong), $\Lambda^0 \to p\pi^-$ (weak) one can measure completely the density matrix of the $Y^\ast(1385)$ (refs. [1, 4, 5]).

2. Polarization parameters and parity conservation

The choice of the frame (helicity or transversity) and the choice of the polarization parameters (density matrix elements or multipole parameters) are a matter of taste and have no fundamental consequence. However, like the authors of AW, we find it convenient to use the transversity multipole parameters. Indeed we have defined [1, 2] a set of real parameters $r_M^{(L)}$ which are proportional to the real or imaginary parts of the $T_M^{(L)}$'s of AW. For a spin-$\frac{3}{2}$ resonance the polarization is described by 15 parameters $r_M^{(L)}$, $L = 1, 2, 3; M = -L, \ldots, +L$. If the resonance is produced in a B-symmetric reaction, in transversity quantization the 8 parameters $r_M^{(L)}$ with $M$ odd vanish [1, 2], and we are left with the 7 parameters: $r_0^{(L)}(L = 1, 2, 3)$ and $r_{\pm 2}^{(L)}(L = 2, 3)$. They are related to the parameters $T_M^{(L)}$ of AW by:

$$r_0^{(L)} = \frac{2}{\sqrt{3}} T_0^{(L)},$$

$$r_2^{(L)} = \sqrt{\frac{3}{2}} \text{Re} T_2^{(L)}, \quad r_{-2}^{(L)} = \sqrt{\frac{3}{2}} \text{Im} T_2^{(L)}.$$ (3a)

Of course, if the 15 polarization parameters of the spin-$\frac{3}{2}$ particle are measurable, as is the case for the $Y^\ast(1385)$, one must verify that the 8 parameters $r_M^{(L)}$ with $M$ odd are zero $^\dagger$. This check has been done in AW, and the result is that these $r_M^{(L)}$'s are compatible with zero to within 2 standard deviations.

3. The polarization domain

For spin-$j$ particles the density matrix $\rho$ is represented by a point in an $N$-dimensional Euclidean space, $N = (2j + 1)^2 - 1$. The coordinates of $\rho$ must satisfy constraints due to the positivity of the density matrix. The polarization domain $\mathcal{D}$ is the domain of definition of the polarization parameters. It has a well defined shape (independently of the choice of coordinates). The main properties of $\mathcal{D}$ are: (i) $\mathcal{D}$ is convex; (ii) its interior represents the density matrices of maximal rank $2j + 1$ (among them is the unpolarized density matrix $\rho_0 = 1/(2j + 1)$); (iii) its boundary $\partial \mathcal{D}$ represents the density matrices of rank $< 2j + 1$.

The polarization degree $d_\rho$ of $\rho \in \mathcal{D}$ is given by the distance between the representative points $\rho$ and $\rho_0$. Its range of values is from 0 (for the unpolarized state)

$^\dagger$ If this is not satisfied, most likely the presence of non-expected $Y_M^{(L)}$ in the decay angular distribution reveals the existence of interference between the resonance channel and the background
to 1 (for the pure states). Hence $D$ is inscribed inside the $N$-dimensional unit sphere $S_N$, centered at $p_0$. The expression of $d_\rho$ in terms of multipole parameters $r^{(L)}_M$ is:

$$d_\rho = \left( \sum_{L,M} (r^{(L)}_M)^2 \right)^{1/2}.$$

(4)

As we recalled previously the density matrix of $B$-symmetric spin-$\frac{3}{2}$ particles is described by 7 non-vanishing parameters $r^{(L)}_M$. The seven-dimensional domain $D$ has been described in ref. [1]. It is the intersection of two quadrics $C_e$, and it is defined by the relations ($e = \pm 1$)

$$(\{r^{(2)}_2 + er^{(3)}_2\}^2 + [r^{(2)}_{-2} + er^{(3)}_{-2}]^2 + [r^{(2)} + e\frac{1}{\sqrt{5}}(2r^{(1)}_0 - r^{(3)}_0)]^2)^{1/2}$$

$\leq \frac{1}{\sqrt{3}} + e\frac{1}{\sqrt{5}}(r^{(1)}_0 + 2r^{(3)}_0).$ (5)

It is interesting to compare the volume of $D$ to that of the unit sphere $S_7$. For this it is convenient to make an orthogonal transformation in the 7-dimensional polarization space.

One defines

$$\xi^{(1)}_e = \frac{1}{\sqrt{2}} [r^{(2)}_0 + e\frac{1}{\sqrt{5}}(2r^{(1)}_0 - r^{(3)}_0)],$$

$$\xi^{(2)}_e = \frac{1}{\sqrt{2}} [r^{(2)}_2 + er^{(3)}_2],$$

$$\xi^{(3)}_e = \frac{1}{\sqrt{2}} [r^{(2)}_{-2} + er^{(3)}_{-2}],$$

$$t = \frac{1}{\sqrt{5}} [r^{(1)}_0 + 2r^{(3)}_0].$$

(6)

In these variables the polarization degree reads

$$d_\rho^2 = t^2 + \xi^2_e + \xi^2_e,$$

(7)

and the equations of the sphere $S_7$ and of the domain $D$ are

$$S_7 : t^2 + \xi^2_e + \xi^2_e = 1,$$

(8)

$$D : -\frac{1}{\sqrt{3}} \leq t \leq \frac{1}{\sqrt{3}}, \quad \xi^2_e \leq \frac{1}{6}(1 + e\sqrt{3}t)^2,$$

(9)

where $\xi^2_e$ is a short notation for $\sum_{i=1}^3 (\xi^{(i)}_e)^2$. 


We emphasize that $D$ is much smaller than $S_7$. Indeed the volume of $D$ is

$$V(D) = \int_{-1/\sqrt{3}}^{+1/\sqrt{3}} \left( \frac{4\pi}{3} \right)^{\frac{2}{3}} \left( \frac{1 + t \sqrt{3}}{\sqrt{6}} \right)^{\frac{3}{2}} \left( \frac{1 - t \sqrt{3}}{\sqrt{6}} \right)^{\frac{3}{2}} dt = \frac{64\pi^2}{8505\sqrt{3}} = 0.04288, \quad (10)$$

while the volume of the interior of $S_7$ is

$$V(S_7) = \frac{\pi^3}{\Gamma\left(\frac{3}{2} + 1\right)} = 4.7248. \quad (11)$$

Hence, the ratio of the volumes is

$$V(S_7)/V(D) = \frac{1}{110.188}. \quad (12)$$

The authors of AW do not verify that their three measured points belong to $D$. They only verify that each parameter $r_M^{(L)}$ is inside the physical bounds which are [6

$$|r_M^{(L)}| \leq \begin{cases} \sqrt{\frac{3}{5}} & \text{for } r_0^{(1)} \text{ and } r_0^{(3)}, \\ \frac{1}{\sqrt{3}} & \text{for the 5 other parameters.} \end{cases} \quad (13)$$

These conditions yield a domain $P$ whose volume is

$$V(P) = \left( \frac{2}{\sqrt{3}} \right)^5 \left( 2\sqrt{\frac{3}{5}} \right)^2 = 4.9267. \quad (14)$$

Note that $P$ is bigger than $S_7$, and is more than one hundred times bigger than $D$: $V(P)/V(D) = 114.898. \quad (15)$

Consequently, if it is advisable to check that $\rho \in P$ and $\rho \in S_7$, it is absolutely necessary to verify that the representative point of the measured polarization belongs to the polarization domain $D$.

4. Tests of the precision and the positivity of a polarization measurement

To estimate the precision of an experimental result $\rho_{\text{exp}} = \{r_M^{(L)}\}$, it is interesting to compare the relative sizes of the statistical $\Delta r_M^{(L)}$ and of the polarization domain $D$. We propose here a quantitative procedure. For given $r_M^{(L)}$ and $\Delta r_M^{(L)}$, the equation

$$\sum_{L,M} \left( \frac{x_M^{(L)} - r_M^{(L)}}{\Delta r_M^{(L)}} \right)^2 = \chi^2 \quad (16)$$

We assume that the systematic errors are negligible. This assumption seems frequently made in experimental papers.
defines an ellipsoid $E_{\chi^2}$ in the $N$ variables $x^{(l)}_M$. For $N$ degrees of freedom, to each value of $\chi^2$ corresponds a confidence level for the points on $E_{\chi^2}$ to be compatible with the experimental point $\rho_{\text{exp}}$. The volume of $E_{\chi^2}$ is

$$V(E_{\chi^2}) = V(S_N) \left( \prod_{L,M} \Delta x^{(l)}_M \right) (\chi^2)^{1/2} N,$$

where $S_N$ is the $N$-dimensional unit sphere.

To have a feeling for the precision of a measure, we choose $\chi^2$ such that the confidence level is $\frac{1}{2}$ (e.g. for $N = 7$, $\chi^2 = 6.346$), we denote by $E(\frac{1}{2})$ the corresponding ellipsoid, and we compute the ratio

$$\lambda = \frac{V(E(\frac{1}{2}))}{V(D)};$$

the smaller this ratio, the better the precision.

Table 1 gives the value of $\lambda$ (computed from eqs. (12), (17) and (18)) for the three points measured by AW (their tables 2, 3 and 4). Of course, when a measurement is made from a small number of events, it is not astonishing that $E(\frac{1}{2})$ could be bigger than $D$ (e.g. for point 1, $\lambda = 7.41$).

Table 1

<table>
<thead>
<tr>
<th>Point</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table of AW</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Number of events</td>
<td>37</td>
<td>320</td>
<td>282</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>7.41</td>
<td>0.10</td>
<td>0.02</td>
</tr>
<tr>
<td>$\mu \leq$</td>
<td>0.003</td>
<td>0.05</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Furthermore we would like to take into account the statistical errors to estimate the positivity of a polarization measurement. For this we consider the intersection $D \cap E(\frac{1}{2})$ of the polarization domain $D$ with the ellipsoid $E(\frac{1}{2})$. The ratio $\mu$ of the volume of $D \cap E(\frac{1}{2})$ to that of $E(\frac{1}{2})$,

$$\mu = \frac{V(D \cap E(\frac{1}{2}))}{V(E(\frac{1}{2}))} \leq 1,$$

gives some quantitative information on the compatibility of the measure with the positivity condition. The bigger the ratio, the better the compatibility. Table 1 gives an upper limit to the ratio $\mu$ (computed by a Monte-Carlo method) for the three points measured by AW.
5. Comparison with theory – Two-dimensional plots

The comparison of an experimental result \( \rho_{\text{exp}} = \{ r^{(L)}_M \} \) (with statistical errors \( \Delta r^{(L)}_M \)) with a theoretical prediction \( \rho_{\text{th}} = \{ x^{(L)}_M \} \), is generally made by computing the \( \chi^2 \) value (from eq. (16)) and the corresponding confidence level for \( N \) degrees of freedom.

We think that this is completely insufficient for polarization measurements. Indeed the \( \chi^2 \) test studies only the compatibility between \( \rho_{\text{exp}} \) and \( \rho_{\text{th}} \). In the usual procedure (eq. (16)), the confidence level improves when the errors increase. In the limit a measurement with infinite errors is perfectly compatible with any theoretical measurement. It is therefore necessary to know first the accuracy of the experiment one cannot forget that the \( r^{(L)}_M \)'s are not free parameters, they are constrained to represent a point \( \rho_{\text{exp}} \) of the polarization domain \( \mathcal{P} \). Hence if the size of the errors and the size of \( \mathcal{P} \) are of the same order of magnitude (e.g. \( \lambda > 0.2 \)), the comparison of \( \rho_{\text{exp}} \) with \( \rho_{\text{th}} \) is not very significant.

![Fig. 1. Two-dimensional plot for the point 1 (table 2 of ref. [3]). The two-plane of the plot is defined by the 3 points \( \rho_0, \rho_{\text{exp}} \) and \( \rho_{\text{th}} \). This figure shows the section by this plane of the physical bounds domain \( \mathcal{P} \), the polarization domain \( \mathcal{P} \) and the two ellipsoids \( \mathcal{E}_{0.07} \) (whose points have a level of confidence \( \geq 0.07 \)) and \( \mathcal{E}_{0.07} \) (which passes through \( \rho_{\text{th}}, \text{eq. (20)} \)).](image-url)
Fig. 2. Two-dimensional plot for the point 2 (table 3 of ref. [3]). The two-plane of the plot is defined by three points $\rho_0$, $\rho_{\exp}$ and $\rho'$ which is the closest to $\rho_{\exp}$ of the set of points predicted by the quark model eq. (21). This figure shows the section by this plane of the physical bounds domain $P$, the polarization domain $D$ and the two ellipsoids $E(\frac{1}{2})$ (whose points have a level of confidence $>\frac{1}{2}$) and $E'$ (which passes through $\rho'$).

The authors of ref. [1] have emphasized the necessity to know the polarization domain $D$ for each experiment and they have proposed a procedure for plotting on it the experimental points with their errors. To visualize easily the position of $\rho_{\exp}$ and $\rho_{\text{th}}$ with respect to $D$ when the dimension $N$ of the polarization space is large, we propose here a procedure using a two-dimensional plot. We draw the intersections of $\mathcal{D}$ and $E(\frac{1}{2})$ with the two-dimensional plane defined by the three points $\rho_0$, $\rho_{\exp}$ and $\rho_{\text{th}}$. For spin $\frac{3}{2}$, the intersections of this two-plane with the quadrics $C_+$ and $C_-$ (eq. (5)) which bound $\mathcal{D}$ are conics, and the intersection with $E(\frac{1}{2})$ is an ellipse.

For reactions (1), because of angular momentum conservation one has rank $\rho = 2$, at any $s$ and $t$ (ref. [1]). Then the representative point is on the boundary $\partial D$ of the polarization domain, at the intersection of the quadrics $C_+$ and $C_-$, eq. (5). More precisely, for these reactions, several simple models (e.g. $\pi$ meson exchange with magnetic coupling [7], SU (6)$_W$ invariance [8], quark model [9]) predict, for any $s$ and $t$

$$\rho_{\text{th}} : x_0^{(2)} = -\frac{1}{\sqrt{3}}, \quad \text{all other } x_M^{(L)} = 0.$$  

(20)

$\dagger$ Of course the two-dimensional plot, which is a section in the $N$-dimensional polarization space, does not contain a complete information on the $N$ measured parameters.
In AW there is one experimental point (point 1) for the two quasi-two-body reactions (1). They find this point compatible with $\rho_{th}$ of eq. (17); the $\chi^2$ is 4.07 and the corresponding confidence level, for 7 degrees of freedom, is 77%. Fig. 1 shows the two-dimensional plot for this point. The experimental point is outside the polarization domain $\mathcal{D}$ but the statistical errors are so large $^+$ (the corresponding $\lambda = 7.41$) that the point $\rho_{exp}$ is compatible with most points of $\mathcal{D}$. This situation shows, as we noticed, that a good $\chi^2$ value is not a sufficient criterium for testing a theory.

For reactions (2), the prediction of the quark model [10] is

$$\rho_{th} : r^{(3)}_{M} = 0, \quad \text{no conditions for } r^{(L)}_{M}, \quad L = 1, 2. \quad (21)$$

Conditions (21) do not define a single theoretical point, they define a four-plane in the seven-dimensional polarization space. We denote by $\rho'$ the orthogonal projection of $\rho_{exp}$ on this four-plane, and the two-dimensional plot for this case is the section of $\mathcal{D}$ and $E(\frac{1}{2})$ by the two-plane defined by $\rho_0, \rho_{exp}, \rho'$. In AW there is one experimental point (points 2 and 3) for each inclusive reaction (2). They give the value of $\chi^2$ for point 2, the corresponding confidence level (3 degrees of freedom) is 28%. For point 3

$^+$ Whatever the statistical errors we notice that the median value $r^{(2)}_{0} = - (2/\sqrt{3}) 0.52$ corresponds to an angular distribution of the decay $Y^{*} \rightarrow \Lambda\pi$ which is not positive definite. See ref. [1b], subsect. 3.2.3 and fig. 6.
the level of confidence would be less than 0.5%. Figs. 2 and 3 show the two-dimensional plots for points 2 and 3. These experimental results are based on a number of events 10 times bigger than that of point 1 and the size of errors is definitely smaller than the size of $\mathcal{D}$.

Although the experimental points are both outside the polarization domain, they are very close to it. This suggests that the true polarization is close to the boundary of $\mathcal{D}$ and not very sensitive to $p_\parallel$ and $p_\perp$ and to the variables which are summed over in the inclusive reaction. Indeed in a convex domain if the barycenter of a set of points is close to a curved boundary, most of the points of the set should be near the barycenter.

Furthermore, we remark that the experimental points, specially point 3, are in disagreement with the quark-model prediction.

6. Conclusion

The representative point of a positive density matrix belongs to a domain $\mathcal{D}$ in the $N$-dimensional polarization space. In this paper, we have emphasized that any analysis of polarization measurement must be done in terms of the polarization domain which is smaller than the domain of the physical bounds. This can be done graphically as proposed in ref. [1]. Here for experimental results given with statistical errors we have defined two parameters $\lambda$ and $\mu$ which estimate the precision of the measure and its compatibility with the positivity condition. Moreover we have shown that the $\chi^2$ test alone is not meaningful for the comparison of an experimental result with a theoretical prediction when the errors are too big. To visualize directly this comparison we have proposed a two-dimensional plot which shows the position of the theoretical and experimental points with respect to $\mathcal{D}$.

To illustrate our discussion we have studied the interesting experimental results of AW (ref. [3]), on the complete density matrix of the $Y^*(1385)$ produced in quasi-two-body and in inclusive reactions. For the case of quasi-two-body reactions we hope to have soon results with better statistics, because of course, it is hopeless to measure precisely 7 parameters from 37 events.

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References