The Demazure–Tits subgroup of a simple Lie group

L. Michel, J. Patera,\textsuperscript{a) and R. T. Sharp\textsuperscript{b)}}

Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France

(Received 5 February 1987; accepted for publication 19 August 1987)

The Demazure–Tits subgroup of a simple Lie group $G$ is the group of invariance of Clebsch–Gordan coefficients tables (assuming an appropriate choice of basis). The structure of the Demazure–Tits subgroups of $A_n$, $B_n$, $C_n$, $D_n$, and $G_2$ is described. Orbits of the permutation action of the DT group in any irreducible finite-dimensional representation space of $A_2$, $C_2$, and $G_2$ are decomposed into the sum of irreducible representations of the DT group.

I. INTRODUCTION

The purpose of this paper is to study a certain finite subgroup of any simple compact Lie group $G$. We call the subgroup the Demazure'–Tits' group and denoted it by DT or DT$(G)$.

The maximal tori (called the Cartan subgroups) of a compact semisimple Lie group $G$ are all conjugate. They are isomorphic to $U(1)^l$, where $l$ is the rank of $G$. The centralizer $C_G(g)$ of $g$ in $G$ contains a Cartan subgroup; the elements $g \in G$, whose centralizer is exactly a Cartan subgroup, are called regular. They form an open dense set in $G$.

Given a Cartan subgroup $H \subseteq G$, one considers its normalizer $N_G(H)$ (the largest $G$ subgroup containing $H$ as an invariant subgroup). The quotient $N_G(H)/H = W(G)$ is the Weyl group of $G$. This is a finite group with a natural action on the Cartan subalgebra $h$ (the Lie algebra of $H$) of $G$ generated by reflections along the simple roots. The importance of the Weyl group in the theory of Lie algebras, Lie groups, and their representations is well recognized. However, the exact sequence

$$1 \rightarrow U(1)^l \rightarrow N_G(U(1)^l) \rightarrow W(G) \rightarrow 1,$$

in general does not split, so $W$ is not a subgroup of $G$, where $G$ is simply connected compact. Among the finite subgroups of the normalizer $N_G(U(1)^l)$ that are mapped by $\partial$ onto $W$ there is a natural one $DT(G)$, defined by (2.15) below, that has been first pointed out by Demazure and Tits.\textsuperscript{2} Its intersection with $U(1)^l$ is the group of square roots of 1, hence it is the extension

$$1 \rightarrow Z^l_2 \rightarrow DT(G) \rightarrow W(G) \rightarrow 1,$$

which is naturally deduced from (1.1).

Physicists' interest in the Demazure–Tits group $DT(G)$ is most likely to originate either from the similarity of its action in representation space to the action of the Weyl group in weight space, or from the fact that it permutes (with some changes of sign) the physical states of a $G$-irreducible space, thus making it possible to keep the same states even without the full Lie group symmetry. It is a finite subgroup of $G$ that preserves the root space decomposition (Cartan decomposition) of the Lie algebra of $G$. The group $DT(G)$ has occasionally appeared in mathematics literature; however, recognition of its usefulness in applied problems relevant to physics is quite recent (cf. Ref. 3, where the group $DT$ is denoted by $N$). A systematic use of $DT(G)$ has been made as the group of invariance of table of the Clebsch–Gordan coefficients (relative to an appropriate basis choice). In computing Clebsch–Gordan coefficients for $G = SU(5)$, $O(10)$, and $E_6$ (cf. Refs. 4–6) $DT$ was used as a group of transformations among CGC of the same values. Practically it allows a small fraction of nonzero CGC to represent all.

In this article we give in Sec. II the structure of $DT(G)$ for the classical groups $A_l, B_l, C_l, D_l$, and for $G_2$. Section III contains some examples of the DT group in lowest representations. In general, it is very interesting to decompose an irreducible $G$-representation space $V_\Lambda$ ($\Lambda$ is the highest weight) into a direct sum of subspaces irreducible with respect to $DT(G)$. For groups $G$ of rank $l = 2$ we describe $DT(G)$ in detail in Secs. IV–VI. Namely, we find its character table, decompose any $V_\Lambda$ into $DT$-invariant subspaces, and identify each $DT$-conjugacy class as a $G$ class of elements of finite order (Sec. VII). The last step opens the possibility of using the powerful computing methods\textsuperscript{7–10} with elements of finite order in $G$ for the study of conjugacy classes of $DT$ in all representations of $G$. The simple Lie group $G$ in this article is always the simply connected one. Section VIII contains a summary of our results and some open problems. The Appendix contains a summation formula, which, as far as we know, does not appear in literature.

We denote a group (finite or continuous) by bold capital letters; for a Lie algebra we use lowercase bold symbols except for groups or algebras of specific types like $A_l$ or $SU(3)$, etc. The symbols $W(g)$ and $W(G)$, $DT(G)$ and $DT(g)$, etc., where $g$ is the Lie algebra of $G$, are used as synonyms.

II. THE STRUCTURE OF THE DEMAZURE–TITS SUBGROUPS OF THE SIMPLE SIMPLY CONNECTED LIE GROUPS

We denote by $(\lambda, \mu)$ the Cartan–Killing positive definite scalar product on the compact semisimple Lie algebra $g$, and let the roots be $\alpha, \in \Delta$, its root system in a chosen Cartan subalgebra $h$; $\Delta$ is the root system of $g$. If $l$ is the rank of $g$
then the Weyl group $W(g)$ is generated by the reflections $r_i$, $i = 1,...,l$, along the simple roots $\alpha_i$,$$
abla \lambda = \lambda - 2 (\alpha_i, \lambda) (\alpha_i, \alpha_i)^{-1} \alpha_i. \quad (2.1)$$When $\lambda$ itself is a simple root, say $\alpha_i$, $$r_i \alpha_i = \alpha_i - \alpha_i A_i, \quad (2.2)$$where$$A_i = 2 (\alpha_i, \alpha_j) (\alpha_j, \alpha_j)^{-1} \quad (2.3)$$are the matrix elements of the Cartan matrix of $g$.

We denote by $l = \dim h$ the rank of $g$. Let $\{r_i, 1 \leq i \leq l\}$ be a minimal set of generators of $W(g)$ (the corresponding simple roots $\alpha_i$ form a base of $h$); this group is completely characterized by the relations$$1 \leq i, j \leq l, \quad (r_i r_j)^{m_{ij}} = I, \quad m_{ij} = 1, \quad 2 \leq m_{ij} = m_{ji} \leq 6. \quad (2.4)$$

Note that $r_i r_j = r_j r_i$ when $m_{ij} = 2$. The list of possible values of $m_{ij}$ was given by Coxeter and is summarized in the Coxeter-Dynkin diagram of $g$. Namely, $m_{ij} = (1 - \theta_{ij}/\pi)^{-1}$, where $\theta_{ij}$ is the angle between $\alpha_i$ and $\alpha_j$; it is 2, 3, 4, or 6 according to whether there are zero, one, two, or three lines joining vertices $i$ and $j$. To specify the structure of $W(g)$, we define first a family of matrix groups (see, e.g., Ref. 11).

A. The groups $G(m, p, n)$

Let $m, p, n$ be integers with $p$ dividing $n$; we denote by $A(m, p, n)$ the group of diagonal $n \times n$ unitary matrices $a$ that satisfy the relations$$a_{ii} = 1, \quad 1 \leq i < n, \quad \det(a)^{mp} = 1. \quad (2.5)$$

Let $\Pi_n$ be the group of $n \times n$ permutation matrices; they have one in each row and each column and zeros elsewhere. It is a faithful representation of $S_n$, the group of permutations of $n$ objects. The determinant of a permutation matrix is $\pm 1$ according to the parity of the permutation. We denote by $G(m, p, n)$ the matrix group generated by the groups $A(m, p, n)$ and $\Pi_n$. Obviously, $G(m, p, n)$ is the semidirect product,

$$G(m, p, n) = A(m, p, n) \ltimes \Pi_n. \quad (2.6)$$

All the matrix groups $G(m, p, n)$, except $G(1, 1, n) = \Pi_n$ and $G(2, 2)$, are irreducible over $C$. The only pair of conjugate groups is $G(4, 4, 2)$ and $G(2, 1, 2)$. For a finite group $G$, we denote by $|G|$ the number of its elements. Then

$$|G(m, p, n)| = m^n p^{-1} n!. \quad (2.7)$$

The linear action of the Weyl group $W(g)$ on the Cartan subalgebra $h$ is represented by

$$W(A) = G(1, 1, 1 + 1), \quad W(B_i) = W(C_i) = G(2, 1, l), \quad W(D) = G(2, 2), \quad W(G_2) = G(6, 2). \quad (2.8)$$

Exceptionally, for $A_i \sim U_{l+1}$, we have used the Cartan algebra of $U_{l+1}$; in it the Cartan algebra of $A_i$ is the hyperplane orthogonal to a vector with all coordinates equal.

For a matrix group $G$ we denote by $S_G$, or sometimes by $G^*$, its unimodular subgroup (i.e., the group of matrices with determinant 1). Note the isomorphism,

$$SG(2, 1, 3) = W(B_3)^* \sim S_4. \quad (2.9)$$

We recall now, at least in a particular case, the definition of the wreath product: given a group $K$, the wreath product by $S_n$, which we denote by $K \wr n$, is the semidirect product

$$K \wr n = K^n \rtimes S_n \quad (2.10)$$

of $S_n$ by $n$ copies of $K$, $S_n$ acting by permutations on the $n$ factors of $K^n$. For a finite group $K$,

$$|K \wr n| = |K|^n n!. \quad (2.11)$$

Let us point out that

$$G(m, 1, n) \sim Z_m \wr n; \quad e.g., \quad W(B_1) \sim Z_2 \wr 1. \quad (2.12)$$

We will need the following properties of Weyl groups. The Lie algebras of types $B_i$ and $C_i$ have roots of two different lengths; the corresponding reflections form two conjugacy classes in $W(B_i) = W(C_i)$ with, respectively, $l$ and $l(l - 1)$ elements. The elements of the conjugacy class with $l$ elements are the reflections of $A(m, 1, l)$. They commute and generate the Abelian group $A(m, 1, l)$. Here $W(D_l)$ is an index 2 subgroup of $W(B_l)$; when $l$ is odd, $W(D_l)$. That is,$$

W(B_1) = W(D_l) \times Z_2 \sim (-I), \quad (2.13)$$

While the Weyl group $W(g)$ is the same for all groups $G$ that have the same Lie algebra $g$, the Demazure-Tits group $DT(G)$ does depend on the choice of $g$; here we consider only simple simply connected compact Lie groups $G$. We use the notation

$$\prod(x, y, v) = x y v \cdots, \quad (2.14)$$

for a product of $n$ factors, alternately $x$ and $y$. Tits defines $DT(G)$ by its generators $q_i$ and their relations

$$1 \leq i < l, \quad q_i^4 = 1, \quad q_i q_j = q_j q_i, \quad \prod(m_{ij} q_i q_j) = \prod(m_{ij} q_j q_i), \quad (2.15a)$$

$$q_i q_j q_i q_j^{-1} = q_j q_i q_j q_i^{-1}. \quad (2.15b)$$

The $q_i$ are the square roots of 1 in the Cartan subgroup, they generate the kernel of $\bar{\theta}$ in Eq. (1.2). The presence of the exponent $2A_{ij}$ in (2.15b) implies that $DT(B_l)$ and $DT(C_l)$ are different although $W(B_l) = W(C_l)$. Since we will use these relations often we give them more explicitly:

$$q_i^4 = 1, \quad q_i q_j = q_j q_i, \quad (E1)$$

$$m_{ij} = 2: \quad q_i q_j = q_j q_i, \quad (E2)$$

$$m_{ij} = 3: \quad q_i q_j = q_j q_i, \quad q_i q_j q_i q_j = q_j q_i q_j q_i^{-1}, \quad (E3)$$

$$m_{ij} = 2k: \quad (q_i q_j)^k = (q_j q_i)^k, \quad q_i q_j q_i q_j^{-1} = q_j q_i q_j q_i^{-1}. \quad (E4)$$

Consider two semisimple Lie groups $G$ and $G'$ both of rank $l$. If the Coxeter-Dynkin diagram of $G$ is a subdiagram of the extended Coxeter-Dynkin diagram of $G'$, then one has for the corresponding $DT$ groups,

$$DT(G) \subset DT(G'). \quad (2.16)$$

Clearly $G$ and $G'$ have the same Cartan subgroup $U_{l+1}$ and $N_G(U_{l+1}) \subset N_G(U_{l+1})$. Since the corresponding $DT$ groups have the same kernel $Z_{2l}$, (2.16) holds. If the rank of $G'$ is lower than $l$, (2.16) still holds provided the Coxeter-Dynkin diagram of $G'$ is a subdiagram of the (nonextended) diagram of $G$. 

Michel, Patera, and Sharp 778
Let $C(G)$ be the center of $G$. The intersection $C(G) \cap DT(G)$ is the group of square roots of $C(G)$. We recall the nature of $C(G)$ in Table I.

B. The DT subgroup of $A_i$

In the natural $(l + 1)$-dimensional representation of $SU_{l+1}$, a Cartan subgroup is represented by diagonal matrices; its subgroup of square roots of the unit is $A(2,2,l+1) - Z_{l+1}$. The Weyl group $-S_{l+1}$ permutes the elements of these diagonal matrices; it can be represented by the group of permutation matrices $\Pi_{l+1}$. The reflections corresponding to permutations of two elements, the $r_i$ corresponding to the permutations of neighboring elements. In $\Pi_{l+1}$, their determinant is $-1$. The unimodular matrices that represent them in $DT(SU_{l+1})$ have been given in Ref. 3 (where they are denoted $R_{ij}$). They are

$$a_i = I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-1},$$

(2.17)

where $I_k$ is the $k \times k$ unit matrix.

Let us introduce the $(l + 1) \times (l + 1)$ diagonal matrices:

$$v_i = -1 \oplus I_l, \quad v_i = I_l \oplus -1 \oplus I_{l+1} = v_i \prod_{k=1}^{l+1} a_k^2, \quad 2 \leq i \leq l+1.$$

(2.18)

They are the reflections of the group $A(2,1,l+1)$ that they generate. For $1 \leq i \leq l$, the matrices $v_i, a_i$ belong to $\Pi_{l+1}$ and generate it since they represent the permutations $(i, l+1)$. Hence we have shown that $v_i$ and the $a_i$'s generate $G(2,1,l+1)$, the unimodular subgroup such that $\Pi_{l+1} CSG(2,1,l+1)$, this shows that the exact sequence (1.2) splits for $l$ even.

$$DT(A_i) = DT(SU_{l+1})^+ \quad (l \text{ even}).$$

(2.20)

When $l$ is even, $\det(-I_{l+1}) = -1$, so we obtain a unimodular representation $\Pi_{l+1}$ of $S_{l+1}$ by multiplying by $-1$ the matrices representing odd permutations; since $\Pi_{l+1} CSG(2,1,l+1)$, this shows that the exact sequence (1.2) splits for $l$ even.

$$DT(A_i) = Z_{l+1} \cap DT(W(A_i)). \quad \text{(l even)}.$$  

(2.21)

This is not the case for odd $l$; e.g., for $l = 1$, $DT(A_1) = Z_2$ (see also at the end of this section). When $l$ is even, we can write explicitly a choice of representatives $a_i$, of the $a_i$'s that realizes the splitting (2.21). We define the $a_i$'s using the sets of indices

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Algebra & $A_i$ & $B_i$ & $C_i$ & $D_i$ \\
\hline
$G$ & $SU_{l+1}$ & Spin$_{2l+1}$ & Spin$_{2l}$ & $D_{2l}$ \\
$C(G)$ & $Z_{l+1}$ & $Z_l$ & $Z_l \times Z_l$ & $Z_l$ (l even) \\
\hline
\end{tabular}
\caption{Structure of the center of a classical simple Lie group $G$.}
\end{table}

\begin{equation}
\mathbf{F}(i,l) = \{k, (0 < k \text{ odd} < i) \cup (i < k \text{ even} < l)\} \quad \text{(2.22a)}
\end{equation}

\begin{equation}
\bar{a}_i = a_l \prod_{j<i} a_k^2. \quad \text{(2.22b)}
\end{equation}

These $\bar{a}_i$ generate a subgroup of $DT(A_i)$ isomorphic to $W(A_i) \sim S_{l+1}$.

The center of $A_i$ is the cyclic group $Z_{l+1}$. When $l$ is odd, the center has a nontrivial square root of unity that is present in every Cartan subalgebra and therefore in $DT(A_i)$. Indeed, the irreducible matrix group $SG(2,1,l+1)$ has a nontrivial center $C(SG(2,1,l+1))$ only when it contains the $-1$ matrix, i.e., for odd $l$. Thus

$$C(DT(A_i)) = 1 \quad \text{or} \quad Z_2(\alpha), \quad \text{for} \ l \ \text{even or odd}.$$  

(2.23)

C. The DT subgroup of $C_i$

Next we consider the $DT$ of the symplectic group $Sp_{2l}$. We denote by $c_i$ the generators of this group. The equations (E) applied to them become

\begin{equation}
c_i' = c_i, \quad c_i' c_j' = c_j c_i' c_i^{-1}, \quad c_i c_{i+1} = c_{i+1} c_i c_i^{-1}, \quad \text{in} \ i < l, \ 
\end{equation}

\begin{equation}
c_i c_{i+1} = c_i c_{i+1} c_{i+1} c_i^{-1} = c_i c_{i+1} c_{i+1}^{-1} c_i c_{i+1}^{-1}, \ 
\end{equation}

\begin{equation}
c_i c_{i+1}^2 = c_i c_{i+1} c_{i+1}^{-1} = c_{i+1} c_i c_i^{-1} c_i = c_i c_{i+1} c_{i+1} c_i^{-1} c_i c_i^{-1} c_i. \ 
\end{equation}

(2.24)

According to (2.16), for $1 \leq i < l$ the $c_i$'s generate $DT(A_{l-1}) \subset DT(C_i)$. In order to complete our study of $C_i$, our strategy is to consider its elements $s_i$, $1 \leq i < l$, "above" the commuting reflections $r_i$ generating $A(2,1,l) \subset W(C_i)$, i.e.,

$$\theta(s_i) = r_i, \quad s_i = c_i, \quad 1 < i < l,$$

(2.25)

$$s_i = u_i s_i u_i^{-1} \quad \text{with} \ u_i = \prod_{k<i} c_k.$$  

(2.26)

(In the II symbol, when the factors do not commute, they are always assumed to be placed in order of increasing index value: $u_i = c_{i} c_{i+1} \cdots c_l c_{l-1}$. We know that these reflections commute among themselves. We now prove the following lemma.

**Lemma 1:** The elements $s_i$ commute among themselves.

We first verify it for $s_{l-1}$ and $s_j$. Indeed from (2.24) and (2.25), we compute

\begin{equation}
s_{l-1} s_j = c_{l-1} c_{l-2} c_{l-1}^{-1} c_l = c_{l-1} c_{l-1} c_{l-1} c_{l-1}^{-1} = s_l s_{l-1}. \ 
\end{equation}

(2.27)

Because $c_i$ and $c_j$ commute when $|i - j| > 1$, with $u_i = \Pi_{k<i}^{-1} c_k$, we have

$$s_i s_j = u_i s_j u_i^{-1} s_i,$$

$$s_{l-1} s_j = u_l s_{l-1} u_l^{-1},$$

$$s_{l-1} s_j = u_l s_{l-1} u_l^{-1} s_{l-1}, \quad \text{for} \ i < l - 1.$$  

(2.27)

We need the relation [use (2.24) twice]
to prove by recourse that \( s_i \) and \( s_{i+1} \) commute. It is true for \( i = l - 2 \):
\[
s_{l-2} s_{l-1} = c_{l-1} s_{l-2} c_{l-2}^{-1} s_{l-1} c_{l-1}^{-1}
\]
\[
= c_{l-1} s_{l-2} c_{l-1}^{-1} s_{l-1} c_{l-1}^{-1}
\]
\[
= c_{l-1} s_{l-2} c_{l-1}^{-1} s_{l-2} = s_{l-1} s_{l-2}.
\]
Assuming that it is true for \( i = k \), we prove it for \( i = k - 1 \),
\[
s_{k-1} s_k = c_k s_{k-1} c_k^{-1} s_k
\]
\[
= c_k c_{k-1} s_k c_{k-1}^{-1} s_{k-1} c_k^{-1}
\]
\[
= c_k c_{k-1} s_k c_{k-1}^{-1} c_k^{-1}
\]
\[
= c_k c_{k-1} s_k c_{k-1}^{-1} c_k^{-1}
\]
\[
= c_k c_{k-1} s_k c_{k-1}^{-1} c_k^{-1}
\]
\[
= c_k s_{k-1} c_k^{-1} s_{k-1} c_k^{-1}
\]
\[
= c_k s_{k-1} c_k^{-1} s_{k-1} c_k^{-1}
\]
\[
= s_k s_{k-1}.
\]
Finally when \( i < j - 2 \), we define as before \( u = u_i u_{j-1}^{-1} \).
Then
\[
s_i s_j = u s_{j-1} u^{-1} s_j
\]
\[
= u s_{j-1} u^{-1} = u s_{j-1} u^{-1} = s_j s_i.
\]
Using (2.24), we find
\[
s_i^2 = \prod_{k=1}^{l} c_i^2
\]
and remark that all the squares are different. Similarly,
\[
c_i^2 = s_i^2 s_i + 1 \Rightarrow c_i^2 = s_i^2 \quad (1 < i < l - 1).
\]
Hence the \( s_i \) commute also with the \( c_i^2 \). They generate an Abelian group containing the kernel in (1.2) of \( DT(C_i) \).
Moreover, the computation of the \( s_i \)'s shows that the covering of \( A(2,1,l) \subset W(C_i) \) in \( DT(C_i) \) is
\[
\theta^{-1}(A(2,1,l)) = Z_i^l.
\]
When \( 1 < i < l - 1 \), we choose other representatives \( \hat{c}_i \) of the \( r_i \)'s,
\[
\theta(\hat{c}_i) = \theta(c_i) = r_i
\]
\[
\hat{c}_i = s_i^2 c_i = c_i s_{i-1}^2 \quad (1 < i < l - 1),
\]
where the last equality is obtained by a repeated use of Eqs. (2.24). We verify that
\[
\hat{c}^2 = 1, \quad (1 < i < l - 2), \quad (\hat{c}, \hat{c}_{i+1})^3 = 1 \quad (1 < i < l - 1).
\]
This shows that \( DT(C_i) \) contains a subgroup isomorphic to \( W(A_{i+i-1}) \sim S_i \). We verify that it acts on the \( s_i \) by permutations
\[
\hat{c}_i s_i \hat{c}_i^{-1} = s_i, \quad \hat{c}_i s_i \hat{c}_i^{-1} = s_{i+1},
\]
\[
\hat{c}_i s_i \hat{c}_i^{-1} = s_i \quad (i < j \text{ or } i > j + 1).
\]
This completes the proof of the isomorphism
\[
DT(C_i) \sim Z_i^l \sim G(4,1,l).
\]
The center, \( C(DT(C_i)) = Z_i(s) \), of this group is the diagonal subgroup of \( Z_i^l \). It is generated by
\[
s = \prod_{k=1}^{l} s_k.
\]
Observe that
\[
C(Sp_{2i}) \cap C(DT(C_i)) = Z_i(s)
\]
where \( \alpha \) has been defined in (2.23),
\[
s^2 = \prod_{k=odd} c_k^2 = \alpha.
\]
The matrices representing \( c_i \)'s in the 2l-dimensional faithful representation of the symplectic group \( C_i \) are shown in Sec. III. All equations of this section can be thus verified.

\section{D. The DT subgroup of B_i}

Let us now consider the DT of Spin_{3l+1}. We denote by \( b_i \) its generators. For \( 1 < i < l - 1 \), like the \( c_i \), these satisfy (2.24) and (E1). But the last line of Eq. (2.24) is replaced by
\[
b_i b_{i+1} b_i b_{i+1} b_i = b_i b_{i+1} b_i b_{i+1} b_i,
\]
\[
b_i b_{i+1} b_i b_{i+1} b_i = b_i b_{i+1} b_i b_{i+1} b_i,
\]
where \( m_i = 2 \) when \( |i-j| > 1 \), so (E2) applies
\[
b_i b_j = b_j b_i \quad (|i-j| > 1).
\]
From these equations we obtain
\[
Z_i(\eta) \subset C(DT(B_i)), \quad \eta = b_i^2.
\]
Here \( Z_i(\eta) \) denotes the \( Z_2 \) group generated by \( \eta \). The group \( Z_i(\eta) \) is exactly \( C(Spin_{3l+1}) \). As we will see later, \( C(DT(B_i)) \) might be larger.
Since \( W(B_i) = W(C_i) \), we follow the same strategy as for the study of \( DT(C_i) \): we introduce the representatives \( t_i \) of the \( l - 1 \) reflections conjugate to \( b_i \),
\[
t_i = b_i, \quad t_i = b_i t_{i+1} b_i^{-1} = b_i t_{i+1} b_i^{-1} \quad (1 < i < l),
\]
where the \( u_i \) are defined as in (2.25). This time we find that the \( t_i \)'s all have the same square,
\[
t_i^2 = \eta, \quad \eta^2 = 1
\]
and, instead of commuting among themselves, we demonstrate that they "anticommutate." More precisely their commutator is \( \eta \),
\[
t_i t_j t_i^{-1} t_j^{-1} = \eta \quad (1 < i, j < l).
\]
For this we follow the same path of computations as in Eq. (2.26)–(2.31):
\[
t_{i-1} t_i = b_i b_{i+1} b_i b_{i+1} b_i^{-1} b_i
\]
\[
= b_{i-1} b_{i+1} b_i b_{i+1} \eta = \eta t_{i-1} t_i.
\]
Replacing the \( s_i \)'s by \( t_i \)'s and (2.26) by (2.48), Eq. (2.27) carries through:
\[
t_i t_i = \eta t_i, \quad (1 < i < l - 2).
\]
Equation (2.28) depends only on (2.24) which is common for both \( DT(C_i) \) and \( DT(B_i) \). It reads for the latter group,
\[
t_i = b_i t_{i+1} t_i b_{i+1}^{-1} \quad (1 < i < l - 1).
\]
To prove by recurrence that \( t_i \) and \( t_{i+1} \) anticommute, we
prove it first for $i = l - 2$. For this we use (2.50), then (2.49),
\[
t_{i-2}t_{i-1} = b_{i-1}t_{i-1}b_{i-1}^{-1} \\
= \eta b_{i-1}t_{i-1}b_{i-1}^{-1} \\
= \eta b_{i-1}t_{i-1}b_{i-1}^{-1}t_{i-2} = \eta t_{i-1}t_{i-2}.
\]
(2.51)

We assume it true for $i + 1$ and prove it for $i$. For this replace the $s$ and $c$’s of (2.30) by $t$ and $b$’s; use (2.51) instead of (2.29). An $\eta$ will appear and this will conclude the proof of (2.47).

The group defined by Eqs. (2.46) and (2.47) is called a Clifford group. It is also called the extra special two-group in mathematics literature. We denote it by $\mathbf{CL}_j$. Its elements are the monomials of the symbolic polynomial $(1 + \eta)\Pi_{i=1}^{l-1} (1 + t_i)$. Thus its order is
\[
|\mathbf{CL}_j| = 2^{l+1} (1 < i < l).
\]
(2.52)

The group $\mathbf{CL}_2$ is the quaternionic group, generated by two $i\alpha_k$, where the $\sigma_k$, $k = 1, 2, 3$, are the three Pauli matrices. We define
\[
t = \prod_{k=1}^{l} t_k.
\]
(2.53)

From Eqs. (2.46) and (2.47) we get
\[
t_i t_j t_{j+i} = t_{j+i}. \quad t_i = t_{j+i} \quad \text{for } i \equiv 0, 1 \pmod{4};
\]
(2.54)

\[
t^2 = 1, \quad \text{for } i \equiv 0, 3 \pmod{4}.
\]
(2.54)

We have seen that in $W(B_l)$, the subgroup $W(A_{l-1})$ generated by the $r_k$’s, $1 < k < l - 1$, acts as the group of permutations $\mathbf{S}_l$ on the $l$ reflections in $A(2, 1, l)$<$W(B_l)$ ($\triangleleft$ reads “invariant subgroup”). The corresponding action of $\alpha_k$, $1 < k < l - 1$, on the $t_i$, will be by permutations modulo elements in $\text{Ker}(\mathbf{DT}(B_l)) = \Pi_{i=1}^{l-1} Z_2(b_i^2)$. By computation we find that this action is only modulo $\eta$; explicitly,
\[
b_i b_i^{-1} t_j = t_j, \quad \eta b_i b_i^{-1} t_j = t_j,
\]
when $j < i$, $j = i$, $j = i + 1$, $j > i + 1$.
(2.55)

This also shows that $\mathbf{CL}_l$<$\mathbf{DT}(B_l)$. Moreover, since the two subgroups $\mathbf{CL}_j$ and $\mathbf{DT}(A_{l-1})$ intersect only on $1$, this proves that
\[
\mathbf{DT}(B_l) \sim \mathbf{CL}_l \otimes \mathbf{DT}(A_{l-1}) \sim \mathbf{CL}_l \otimes \mathbf{SG}(2, 1, l),
\]
(2.56)

with the action defined in (2.55). From this equation we obtain the action of the $b_i$’s on $t$ defined in (2.53); it is trivial:
\[
b_i b_j^{-1} t = t_i + t_j.
\]
(2.57)

From (2.54), we see that when $l$ is odd, $\eta \in \mathbf{C}(\mathbf{DT}(B_l))$. Finally, with (2.54) we obtain
\[
\mathbf{C}(\mathbf{DT}(B_l)) = Z_2(\eta), \quad Z_4(t), \quad Z_2(\eta) \times Z_2(t),
\]
\[
l \equiv 0, 2, 1, 3,
\]
(2.58)

We recall that for all values of $l$, $\mathbf{C}(B_l) = Z_2(\eta)$.

In Sec. III we give an explicit representation of the $b_i$’s in the 2l-dimensional faithful representation of Spin$_{2l+1}$. We denote by $\varphi$ the homomorphism from Spin$_{2l+1}$ onto SO$_{2l+1}$.$\sim$Spin$_{2l+1}$,$\sim$Z$_2(\eta)$. These two groups are the images of the nontrivial irreducible representations of $B_l$. In the tensorial representations, $\mathbf{DT}(B_l)$ is represented by the splitting image
\[
\varphi(\mathbf{DT}(B_l)) = Z_2(\eta) \times W(B_l) \sim (Z_2(\eta) \times Z_2) \otimes \mathbf{S}_l.
\]
(2.59)

**E. The DT subgroup of $D_l$**

We denote by $d_i$ the generators of $\mathbf{DT}(D_l)$,$\sim$Spin$_{2l}$. Since $D_l = \text{Spin}_{2l}$ is a maximal subgroup of $B_l = \text{Spin}_{2l+1}$ with the same rank $l$, we know from (2.16) that
\[
\mathbf{DT}(B_l) \subset \mathbf{DT}(D_l),
\]
(2.60)

and that it is of index 2, i.e., the same as $W(B_l)$ in $W(B_l)$, since we pass from the latter group to the former one by replacing $A(2, 1, l)$ by its subgroup of unimodular matrices $A(2, 2, l) = \mathbf{SA}(2, 1, l)$. It contains only the products of an even number of reflections $r_i$. We will write the generators $w_i$ of $\mathbf{DT}(A_{l-1})$ as products of pairs of the $t_i$’s. More generally, it follows from the structure of $W$ that we can write the generators of $\mathbf{DT}(D_l)$ in terms of those of $\mathbf{DT}(B_l)$. Namely,
\[
d_k = b_k, \quad d_i = b_i b_{i-1}^{-1} (1 < i < l - 1).
\]
(2.61)

We can verify that the $d_i$’s satisfy the equations corresponding to (E2), and (E3). In particular,
\[
d_{i-1} d_i = d_i d_{i-1}.
\]
(2.62)

Since $\eta \in \mathbf{C}(\mathbf{DT}(B_l))$, it is also in $\mathbf{C}(\mathbf{DT}(D_l))$. It can now be defined by
\[
\eta = d_{i-1} d_i.
\]
(2.63)

We can choose for the generators of $\mathbf{SA}(2, 1, l)$,
\[
w_{i} = t_i t_{i+1} d_{i-1} d_{i+1},
\]
(2.64)

\[
v_i = \prod_{j=i}^{i+1} d_k \quad (1 < i < l - 2).
\]

From Eqs. (2.46) and (2.47) we find immediately that the $l - 1$ $w_i$’s satisfy the same equations so they generate a subgroup $\sim \mathbf{CL}_{l-1}$. This is an invariant subgroup of $\mathbf{DT}(D_l)$ that has a trivial intersection with the subgroup $\mathbf{DT}(A_{l-1})$. These two subgroups generate $\mathbf{DT}(D_l)$. Hence
\[
\mathbf{DT}(D_l) = \mathbf{CL}_{l-1} \otimes \mathbf{DT}(A_{l-1}),
\]
(2.65)

where the action of the $d_i$’s on the $w_i$’s is defined implicitly by (2.55) when the $d_i$’s and the $w_i$’s are expressed, respectively, as functions of $t_i$ and $t_j$ [see (2.61) and (2.64)].

Let us now consider the center of $\mathbf{DT}(D_l)$. As in (2.53) we define
\[
w = \prod_{k=1}^{l-1} w_k = t_i, \quad \text{for } l \text{ even},
\]
\[
\eta t_i = n t_i, \quad \text{for } l \text{ odd}.
\]
(2.66)

Similarly to (2.54) we obtain
\[
w w_i = w_i w, \quad w^2 = 1, \quad \text{for } i \equiv 0, 1 \pmod{4},
\]
\[
w^2 = \eta, \quad \text{for } i \equiv 2, 3 \pmod{4}.
\]
(2.67)

When $l$ is even,
\[
\alpha = \prod_{k=1}^{l} d_k^2,
\]
(2.68)

already defined in (2.23), is in $\mathbf{C}(\mathbf{DT}(A_{l-1}))$. It anticommutes with $t_i$, so it commutes with $d_i$. Hence it is in the
TABLE II. Structure of the center of the Demazure–Tits subgroup of the simple Lie group $D_l$ and its intersection with the center of the Lie group. $Z_n(y)$ denotes a cyclic group generated by $y$.

<table>
<thead>
<tr>
<th>$l$ (mod 4)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{CD}(D_l)$</td>
<td>$Z_n(\alpha) \times Z_n(\eta) \times Z_n(w)$</td>
<td>$Z_n(\eta) \times Z_n(w)$</td>
<td>$Z_n(\alpha) \times Z_n(w)$</td>
<td>$Z_n(w)$</td>
</tr>
<tr>
<td>$\text{CD}(D_l) \cap (D_l)$</td>
<td>$Z_n^2$</td>
<td>$Z_n$</td>
<td>$Z_n(\alpha)$</td>
<td>$Z_n(\eta)$</td>
</tr>
</tbody>
</table>

center of $\text{DT}(D_l)$. We summarize the description of the center of $\text{DT}(D_l)$ and its intersection with the center of $G$ in Table II.

For $l$ even, there are no faithful irreducible representations of $D_l$. We denote again by $\varphi$ the homomorphism from $\text{Spin}_{2l}$ onto $\text{SO}_{2l} \sim \text{Spin}_{2l+1}/Z_2^l$. In the tensorial representations, $\varphi(\text{DT}(B_{2l}))$ is represented by the splitting image,

$$
\varphi(\text{DT}(B_{2l})) = Z_{2l}^{-1} \otimes W(\text{DT}(B_{2l})) \sim (Z_{2l}^{-1} \otimes Z_{2l}^{-1}) \otimes \mathbb{S}_l

$$

(2.69)

F. The DT subgroup of $G_2$

The Weyl group of $G_2$ is the dihedral group of 12 elements isomorphic to $S_3 \times Z_2$. Therefore the order of $\text{DT}(G_2)$ is 48. From (2.16) we know that $\text{DT}(SU_3)$ $\subset \text{DT}(G_2)$ and it has index 2. Note that $\text{DT}(SU_3)$ is isomorphic to $S_3$, [see (2.20) and (2.9)] so that complete. It means that it has no center and no outer automorphism. Hence from a known theorem it has the isomorphism

$\text{DT}(G_2) \sim S_3 \times Z_2$. (2.70)

We have seen that $\text{DT}(A_2) \sim Z_2^2 \times S_3 \sim S_4$ splits. Since $W(G_2) = S_3 \times Z_2$, (2.70) implies that $\text{DT}(G_2)$ also splits,

$\text{DT}(G_2) = Z_2^2 \otimes W(\text{DT}(G_2)) \sim Z_2^2 \otimes S_3 \times Z_2$. (2.71)

We recall that $\text{C}(G_2) = 1$; however, $\text{CD}(G_2) \sim Z_2$

In his paper Tits asks the question: What is the smallest subgroup $W$ of $\text{DT}(G)$ that covers $W(G)$, i.e., $\vartheta(W) = W(G)$? With the knowledge of the explicit structure of the $\text{DT}(G)$ groups we can give the answer. It is found in Table III.

To end this section we summarize in Table IV the information obtained on the structure of the $\text{DT}(G)$ and their centers.

TABLE III. The smallest subgroups of the Demazure–Tits group $\text{DT}(G)$ covering the Weyl group $W(G)$. $K = \ker \vartheta$. The exception for $\text{DT}(A_3)$ is due to the solvability of $S_3 \sim Z_3 \times S_3$. The result can be understood from $A_1 \sim D_5$.

<table>
<thead>
<tr>
<th>$G$</th>
<th>rank $l$</th>
<th>$W$</th>
<th>$W \cap K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_l$</td>
<td>/ even</td>
<td>$-W$</td>
<td>1</td>
</tr>
<tr>
<td>$A_l$</td>
<td>/ odd</td>
<td>$\text{DT}(A_l)$</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>$C_l$</td>
<td>$l = 3$</td>
<td>$\text{CL}_l(S_3)$</td>
<td>$Z_2(\alpha)$</td>
</tr>
<tr>
<td>$C_l$</td>
<td>$l = 3$</td>
<td>$\text{DT}(C_l)$</td>
<td>$Z_2(\alpha)$</td>
</tr>
<tr>
<td>$B_l$</td>
<td></td>
<td>$\text{CL}_l(S_3)$</td>
<td>$Z_2(\eta)$</td>
</tr>
<tr>
<td>$D_l$</td>
<td></td>
<td>$\text{CL}_l(S_3)$</td>
<td>$Z_2(\eta)$</td>
</tr>
<tr>
<td>$G_l$</td>
<td></td>
<td>$-W$</td>
<td>1</td>
</tr>
</tbody>
</table>

III. REPRESENTATIONS OF THE DEMAZURE–TITS GROUPS AND EXAMPLES

Let us underline some common features as well as differences between the well-known group $\text{W}(\text{G})$ and the group $\text{DT}(\text{G})$ that are used subsequently and provide some examples of elements $R_i$, $i = 1, ..., l$, generating $\text{DT}(\text{G})$ in some low-dimensional representations of $G$ of several types and many ranks. The rank $l = 2$ cases are studied in much greater detail in Secs. IV–VI. Other properties of $\text{DT}(\text{G})$ can be found in Sec. III of Ref. 3.

The fundamental weights $\omega_1, ..., \omega_l$ are defined by

$$
(\alpha_i, \omega_k) = \delta_{ik}(\alpha_i, \alpha_i)/2

$$

(3.1)

The weight lattice $Q$ is the $Z$-span of the fundamental weights of $G$,

$$
Q = \{\mu = (a_1, ..., a_l) | \mu = a_1\omega_1 + ... + a_l\omega_l, a_i \in \mathbb{Z}\}

$$

(3.2)

The sector of $Q$ containing only dominant weights (all $a_i > 0$) is denoted $Q^+$. Each orbit of $\text{W}in Q$ is a set of weights that contains precisely one dominant weight, say $\lambda^+$. By definition, the set of lattice points

$$
O(\lambda^+) = \{\mu | \mu = a_\lambda^+, a \in \mathbb{W}\}

$$

(3.3)

is a $\text{W}$ orbit, it is $\text{W}$ invariant and is usually specified by its dominant weight $\lambda^+$. Subsequently, when no ambiguity could arise, we often use $\lambda^+$ for $O(\lambda^+)$; similarly $O(\lambda^+)$ is often denoted by $W\lambda^+$. The number of elements of $O(\lambda^+)$ is equal to the ratio

$$
|O(\lambda^+)| = |W\lambda^+| = |W|/|\text{Stab}_W\lambda^+|

$$

(3.4)

of the order of $W$ to the order of the stabilizer of $\lambda^+$ in $W$. It is tabulated in Ref. 13.

$$
\text{Stab}_W\lambda^+ = \{w | w\lambda^+ = \lambda^+, w \in \mathbb{W}\}

$$

(3.5)

Stab$_W\lambda^+$ is the Weyl group of $\lambda^+$ (semisimple) Lie algebra obtained easily as follows. Take the Coxeter–Dynkin diagram of $W$ (it is the Weyl group of $G$) and attach the coordinates of the dominant weight $\lambda^+$ in the basis of the fundamental weights to the corresponding nodes of the Coxeter–Dynkin diagram. Remove nodes with nonzero coordinates. What remains is the diagram of a semisimple Lie subgroup of $G$ whose Weyl group is Stab$_W\lambda^+$.

An irreducible representation is specified up to $G$ conjugacy by its highest weight $\lambda^+$. Therefore a representation is usually denoted by $\lambda$. An efficient algorithm for finding all $\lambda^+$ in $\Omega(\lambda)$ is given in Refs. 12 and 13. For most cases of interest, $\lambda^+$ have been tabulated in Ref. 13 together with the multiplicity of their occurrences in $\Omega(\lambda)$.

The weight system $\Omega(\lambda)$ of a representation $\lambda$ is in-
TABLE IV. Structure of the Demazure–Tits subgroups of simple Lie groups. Symbols $\alpha$, $\beta$, $\gamma$, $\delta$, $\omega$, are, respectively, defined by the following equations: $\alpha$: (2.23), $\beta$: (2.41), $\gamma$: (2.39), $\delta$: (2.44), $\omega$: (2.53), $\omega$: (2.66). Here $Z_\gamma$ denotes a cyclic group of order $n$ generated by $y$. The Clifford group $\text{Cl}_L$ is defined by (2.53) and (2.54).

<table>
<thead>
<tr>
<th>$G$</th>
<th>$l \mod 4$</th>
<th>$\text{D}T(G)$</th>
<th>$\text{C}(\text{D}T(G))$</th>
<th>$\text{C}(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_l$</td>
<td>0, 2</td>
<td>$Z_2 \ltimes S_{l+1}$</td>
<td>1</td>
<td>$Z_2 \ltimes 1$</td>
</tr>
<tr>
<td></td>
<td>1, 3</td>
<td>$-W(B_{l+1})^*$</td>
<td>$Z_2(\alpha)$</td>
<td>$Z_2(\alpha)$</td>
</tr>
<tr>
<td>$B_l$</td>
<td>0, 2</td>
<td>$\text{Cl}<em>L \ltimes \text{D}T(A</em>{l-1})$</td>
<td>$Z_2(\gamma)$</td>
<td>$Z_2(\gamma)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\text{Cl}<em>L \ltimes (Z_2 \ltimes S</em>{l+1})$</td>
<td>$Z_2(\tau)$</td>
<td>$Z_2(\tau)$</td>
</tr>
<tr>
<td>$C_l$</td>
<td>0</td>
<td>$Z_2(\tau)$</td>
<td>$Z_2(\tau)$</td>
<td>$Z_2(\tau)$</td>
</tr>
<tr>
<td>$D_l$</td>
<td>0</td>
<td>$\text{Cl}<em>L \ltimes (Z_2 \ltimes S</em>{l+1})$</td>
<td>$Z_2(\tau)$</td>
<td>$Z_2(\tau)$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$\text{Cl}<em>L \ltimes (Z_2 \ltimes S</em>{l+1})$</td>
<td>$Z_2(\tau)$</td>
<td>$Z_2(\tau)$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$S_3 \times Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
</tr>
</tbody>
</table>

variant under $W$ and decomposes into several $W$ orbits $O(\lambda^+) = O(W\lambda^+)$:

$$\Omega(\Lambda) = \lambda^+ \ast O(\lambda^+) = \Omega(\lambda^+).$$

(3.6)

The same orbit $O(\lambda^+)$ often occurs with multiplicity $\text{mult}_\Lambda(\lambda^+) > 1$ in $\Omega(\Lambda)$. We use $n$ for the multiplicity $\text{mult}_\Lambda(\lambda^+)$ of $\lambda^+$ in $\Omega(\Lambda)$ whenever there is no ambiguity as to what $\Lambda$ and $\lambda^+$. Are. The orbit $O(\Lambda)$ of the highest weight $\Lambda$ is always unique in $\Omega(\Lambda)$, i.e., $\text{mult}_\Lambda(\Lambda) = 1$.

Consider the representation space $V_\Lambda$ and its decomposition

$$V_\Lambda = \lambda^+ \ast \text{D}T(\Lambda) \ast \text{D}T(\Lambda) \ast \lambda^+ \ast O(\lambda^+) = \lambda^+ \ast \lambda^+ \ast O(\lambda^+) \ast O(\lambda^+).$$

(3.7)

Parallel to the decomposition (3.6) of $\Omega(\Lambda)$, where the subspace $V_\Lambda(\lambda^+)$ corresponds to $O(\lambda^+)$. Indeed $V_\Lambda(\lambda^+)$ is the direct sum of weight subspaces $V_\Lambda(\mu)$, $\mu \in O(\lambda^+)$. The dimensions are given by

$$\dim V_\Lambda(\lambda^+) = |W\lambda^+| \dim V_\Lambda(\mu)$$

$$= |W\lambda^+| \text{mult}_\Lambda(\lambda^+).$$

(3.8)

The permutation of weights $\mu = r_1 \mu_1$, $\mu u \in \Omega(\Lambda)$, $r_1 \in W$, by $r_i$'s of (2.1) exactly corresponds to the permutation of weight subspaces $V_\Lambda(\mu)$ by the elements $R_i \in \text{D}T$. Namely,

$$R_i V_\Lambda(\mu) = V_\Lambda(r_i \mu) = V_\Lambda(\mu'), \quad R_i \in \text{D}T, \quad 1 \leq i \leq l.$$ (3.9)

In Ref. 3 the elements $R_i$ are called charge conjugation operators. In practice one is more interested in the transformation properties of individual vectors $v_\mu \in V_\Lambda(\mu)$,

$$R_i v_\mu = v_{r_i \mu}, \quad v_\mu \in V_\Lambda(\mu), \quad v_{r_i \mu} \in V_\Lambda(r_i \mu),$$ (3.10)

rather than in (3.9). Since there may be $n$, $n \geq 0$, linearly independent vectors $v_\mu$, it turns out that the action of $R_i$ on $V_\Lambda(\mu)$ is quite nontrivial even if $r_i$ acts trivially on $\mu$, i.e., if $r_i \mu = \mu$. Although one still has (3.9), it does not imply that $v_\mu = v_\tau$. For examples see Ref. 3 and Appendix C of Ref. 14.

It follows from (3.9) and (3.10) that one can write symbolically

$$\text{D}T V_\Lambda(\lambda^+) = V_\Lambda(\lambda^+) = \sum m_i V(\Gamma_i), \quad m_i \in Z_{>0}. (3.11)$$

The action of $\text{D}T$ is necessarily reducible in subspaces $V_\Lambda(\lambda^+) \ast V_\Lambda$. Indeed, $\text{D}T$, being a finite group, has finitely many irreducible representations $F_i$, $i = 1, 2, \ldots, k < \infty$, while the dimension of $V_\Lambda(\lambda^+)$ has no upper limit; it grows with $\Lambda$. The summation in (3.11) extends over the irreducible representations of $\text{D}T$.

Before turning to specific examples let us recall some notations and conventions. Consider $l$ isomorphic copies of the complex Lie algebra $\text{sl}(2, C)$, $1 \leq i \leq l$, in 1–1 correspondence with the simple roots of $G$. The basis elements $e_i, f_i, h_i$ of each $\text{sl}(2, C)$, are chosen to satisfy

$$[e_i, f_j] = h_i, \quad [h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i, \quad 1 \leq i \leq l.$$

(3.12)

The generator of $G$ can be written as linear combinations of $e_i - f_i$ and $\sqrt{-1}(f_i + e_i)$ for $i \in \{1, \ldots, l\}$ and their commutators. Since we make no direct use of these other generators, there is no need to write them down here. However, we always assume that a Chevalley basis of $G$ has been chosen. It amounts to having the structure constants integer.

The charge conjugation operators $R_i \in G$ can be written as

$$R_i = \exp(f_i) \exp(-e_i) \exp(f_i), \quad 1 \leq i \leq l.$$ (3.13)

They generate the Demazure–Tits group $\text{DT}$. It has been shown in Ref. 3 that

$$R_i = 1, \quad R_i v_\lambda = (-1)^{\langle \lambda, \Lambda \rangle / 2} v_{-\lambda}, \quad v_\lambda \in V_\Lambda(\lambda),$$ (3.14)

where $\Lambda = \text{twice the angular momentum}$ denotes the irreducible representation $A_l$ of dimension $l + 1$ and $\lambda$ is a weight of its weight system $\Omega(\Lambda) = (\Lambda, \Lambda - 2, \ldots, -\lambda)$.

Let us consider examples of $R_i$ in the lowest representations of simple Lie groups of different types.

$(A_l)$ The faithful representation $\Lambda = (100 \cdots 0)$ of dimension $l + 1$.
\[ R_i = I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-i}, \quad 1 \leq i \leq l. \]  

(3.15)

Here \( I_k \) is the \( k \times k \) identity matrix. In matrixlike symbols we write negative signs over the digits.

\( (B_i) \) The matrices \( R_i, 1 \leq i \leq l - 1 \) (denoted by \( b_i \) in Sec. II) corresponding to \( r_i \in \mathbb{W} \) in the (faithful) \( 2^l \)-dimensional spinor representation of \( \text{Spin}_{2l+1} \) are

\[ R_i = (\oplus I_{l-i} \oplus P \oplus (\oplus I_{l-i})), \quad 1 \leq i \leq l - 1, \]

\[ R_l = (\oplus I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-1}), \]  

(3.16)

where \( P \) is the matrix

\[
P = \frac{1}{2}(I_2 \otimes I_2 + \sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_2 - i\sigma_2 \otimes \sigma_1)
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

In particular, one has for \( l = 3 \) the \( B_i \) representation of dimension \( 2^3 \) in a direct sum form, as

\[ R_1 = I_2 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_2, \]

\[ R_2 = I_1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_2 \oplus I_1, \]

\[ R_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

(3.17)

Similarly one has the \( B_i \) representation of dimension \( 2^{l+1} \) that is not faithful (trivial center),

\[ R_i = I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{2l-2l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-1}, \]

\[ 1 \leq i \leq l - 1, \]

\[ R_l = I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-1}, \quad l \geq 2. \]  

(3.18)

\( (C_i) \) Representation of dimension \( 2^l \)

\[ R_i = I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{2l-2l-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-1}, \]

\[ R_i = I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-1}. \]  

(3.19)

Note that, for \( l = 2, B_i \) is identical to \( C_2 \) up to a renumbering \( \alpha_i \rightarrow \alpha_2 \) of simple roots. In this case \( 3.18 \) and \( 3.19 \) refer to the same group in representations of dimension 5 and 4, respectively.

\( (D_i) \) When \( l \) is even no irreducible representation of \( D_i = \text{Spin}_{2l} \) is faithful because the center is not cyclic, \( C(D_i) = Z_2 \). In order to have a faithful representation one can consider the direct sum of the two \( 2^{l-1} \)-dimensional spinor representations. It can be obtained from the \( 2^l \)-dimensional representation of \( B_i = \text{Spin}_{2l+1} \). The matrices \( R_i \) corresponding to \( r_i \in \mathbb{W} \) are

\[ R_i \text{ as in (3.16)}, \quad \text{for } 1 \leq i \leq l - 1, \]

\[ R_l = (\oplus I_{l-1} \oplus Q, \]

with

\[
Q = \frac{1}{2}(I_2 \otimes I_2 - \sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_2 - i\sigma_2 \otimes \sigma_1)
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The \( D_i \) representation of dimension \( 2^l \) has

\[ R_i = I_{l-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{2l-2l-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-1}, \]

\[ R_l = I_{l-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{l-2}. \]  

(3.21)

Somewhat special is the case \( l = 4 \). There are three representations of dimension 8. They differ by the following permutations of \( R_i \)’s,

\[ 1000 \quad \text{as in Eq. (3.21)}, \]

\[ 0001 \quad R_1 \leftrightarrow R_4, \]

\[ 0010 \quad R_2 \leftrightarrow R_3. \]  

(3.22)

\( (G_i) \) Representation of dimension 7,

\[ R_i = I_1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_1, \]

\[ R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

(3.23)

IV. THE DEMAZEUR–TITS SUBGROUP OF \( A_2 \)

In Secs. IV–VI we consider each of the simple Lie groups of rank 2. The description of the Demazure–Tits group DT in these cases is carried much further than for higher ranks because one may expect that the lowest ranks will be used most frequently; also, the derivations and results are simpler. Our analysis serves as a model of what can be learned, at least in principle, about each case, besides being a particularly useful illustration.

Each of the three groups is specified up to an isomorphism by its simple roots \( \alpha_i \) and \( \alpha_2 \), or, equivalently, by the Cartan matrix

\[ (A_2) = \frac{2(\alpha_i, \alpha_j)}{2(\alpha_i, \alpha_i)} = \begin{pmatrix} 2 & -A \\ B & -2 \end{pmatrix}, \]

where

\[ A = B = 1, \quad \text{for } A_2, \]

\[ A = 2B = 2, \quad \text{for } B_2, \]

\[ A = 3B = 3, \quad \text{for } G_2. \]  

(4.1)
The Weyl group $\mathbf{W}$ acts on the weight lattice $Q$, which is the $\mathbb{Z}$ span of two fundamental weights $\omega_1$ and $\omega_2$. In particular,

$$\alpha_1 = 2\omega_1 - A\omega_2, \quad \alpha_2 = -B\omega_1 + 2\omega_2,$$

and therefore

$$\omega_1 = \{1/(4 - AB)\}(2\alpha_1 + A\alpha_2),$$

$$\omega_2 = \{1/(4 - AB)\}(B\alpha_1 + 2\alpha_2).$$

The elements $r_1$ and $r_2$ generate $\mathbf{W}$ by their action (2.1) on the weights $\mu = a\omega_1 + b\omega_2 = (a,b) \in Q$, where $a,b \in \mathbb{Z}$. Namely,

$$r_1(a,b) = (-a, b + Ba), \quad r_2(a,b) = (a + Bb, b).$$

In particular, one has for the simple roots, $r_1\alpha_1 = r_1(2, -A) = (-2, A) = -\alpha_1$, $r_2\alpha_2 = r_2(-B, 2) = (B, -2) = -\alpha_2$. A weight is called dominant if $a, b \geq 0$.

The “lifting” of the action of $\mathbf{W}$ on $Q$ to the action of $\mathbf{DT}$ on $\mathcal{V}_\lambda$, i.e., the homomorphism $\mathbf{DT} \rightarrow \mathbf{W}$, can be set up in several equivalent but not identical ways. To avoid possible ambiguities, we adopt from now on the following prescription. The elementary reflections $r_1, r_2 \in \mathbf{W}$ of (3.1) are lifted into $R_1, R_2$ as given in (3.13) and (3.14). Any other $w \in \mathbf{W}$ is expressed as a word $r_{i_1}r_{i_2} \cdots$ of minimal length in elementary reflections. Then as it is lifted we take the result to be $R_{i_1}R_{i_2} \cdots$. The group $\mathbf{G}$ also contains one element (opposite involution) of maximal length $k_{\text{max}} = \text{number of positive roots of } \mathbf{G}$.

The decomposition of $V_w(\lambda^+) \rightarrow \mathbf{DT}$-irreducible subspaces in the three cases of rank 2 is the main problem solved in the rest of this article. Our task is to find the multiplicities $m_i$ of occurrence of the subspaces $V(\Gamma_i)$, irreducible with respect to the representations $\Gamma_i$ of $\mathbf{DT}$ in the direct sum [cf. (3.11)],

$$V_w(\lambda^+) = \oplus_i m_i V(\Gamma_i), \quad m_i \in \mathbb{Z}_{\geq 0}.$$  

Unlike the $\mathbf{W}$ orbit $\Omega(\lambda^+)$, which is independent of the rest of a weight system $\Omega(\Delta)$ to which it may belong, the decomposition (4.6) depends on $\Delta$ and the multiplicity $n = \text{mult}_\lambda \lambda^+$. For simplicity of notation we write (4.6) as

$$\lambda^+ = \oplus_i m_i \Gamma_i.$$  

Let us now turn to the particular case of the Lie algebra $A_2$ [or Lie group $\text{SU}(3)$]. The multiplicity $n$ of a dominant weight $\lambda^+ = (a,b)$ in an $\text{SU}(3)$ representation $\Lambda = (p,q)$ is the coefficient of the term $P^p Q^q A^a B^b$ in the power expansion of the generating function $^3$

$$\frac{1}{(1 - P)(1 - QA)(1 - QB)(1 - P^2 B)} 
= \frac{1}{(1 - PA)(1 - QA)(1 - Q^2 A)} + \frac{1}{(1 - PA)(1 - QB)(1 - Q^2 A)} + \frac{1}{(1 - PA)(1 - P^2 B)(1 - P^2)} + \frac{1}{(1 - QB)(1 - Q^2 B)(1 - Q^2)}.$$  

From (4.7) we deduce that $n = 0$ unless $p - q + b - a = 0$

<table>
<thead>
<tr>
<th>Class</th>
<th>1</th>
<th>3</th>
<th>2</th>
<th>Number of elements</th>
<th>IR</th>
<th>Representative element</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$r_1$</td>
<td>$r_1r_2$</td>
<td>$c_1$</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>$c_2$</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>$c_3$</td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>$c_4$</td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>3</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>$c_5$</td>
</tr>
</tbody>
</table>

The character table of the $\mathbf{DT}(A_2)$ and $\mathbf{W}(A_2)$ groups. Subscript of the class symbol indicates the order of its elements. EFO denotes the conjugacy class in $\text{SU}(3)$ and IR means irreducible representation.

![Fig. 1. Action of representative elements of conjugacy classes of the Weyl group of $A_2$, on weights of a generic orbit.](image)

<table>
<thead>
<tr>
<th>Demazure - Tits group</th>
<th>$\mathbf{G}$</th>
<th>$\mathbf{DT}$</th>
<th>$\mathbf{SU}(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{EFO}$</td>
<td>100</td>
<td>011</td>
<td>011</td>
</tr>
<tr>
<td>Number of elements</td>
<td>1 3 6 5 8 6</td>
<td>3 6 5 8 6 3</td>
<td>3 6 5 8 6 3</td>
</tr>
<tr>
<td>Class</td>
<td>$r_1$</td>
<td>${r_2, r_3}$</td>
<td>${r_1, r_2}$</td>
</tr>
</tbody>
</table>

The four expressions in the minimum symbol arise, respectively, from terms 4, 3, 2, 1 in (4.7); there is no overlap (i.e., for given $p, q, a, b$ at most one term contributes, namely the one giving the smallest value).

The Weyl group of $A_2$ is isomorphic to $S_3$, the group of permutations of three objects. It is also the dihedral group $D_3$. Its character table is given in Table V. That table contains as well the characters of the $\mathbf{DT}(A_2)$ group, the homomorphism between the classes of elements of $\mathbf{W}$ and $\mathbf{DT}$ groups, and the $\text{SU}(3)$-conjugacy classes of elements of $\mathbf{DT}$.

The character values afforded by the three conjugacy classes of $\mathbf{W}$ are easily deduced using the action of representative elements on the points of a generic orbit $(a,b)$, illustrated on Fig. 1.

The decomposition of Weyl group orbits on the $A_2$ weight lattice into direct sums of irreducible representations of $\mathbf{W}$ is presented in Table VI.

The structure of the Demazure–Tits subgroup $\mathbf{DT}$ of $\text{SU}(3)$ is found either from the $\text{SU}(n)$ case of Sec. II or by a direct computation. It turns out to be the octahedral

$$\{p, q \in \mathbb{Z}, (2p + q \geq 2a + b, p + 2q \geq a + 2b) \}.$$  

(4.8)

The four expressions in the minimum symbol arise, respectively, from terms 4, 3, 2, 1 in (4.7); there is no overlap (i.e., for given $p, q, a, b$ at most one term contributes, namely the one giving the smallest value).

The Weyl group of $A_2$ is isomorphic to $S_3$, the group of permutations of three objects. It is also the dihedral group $D_3$. Its character table is given in Table V. That table contains as well the characters of the $\mathbf{DT}(A_2)$ group, the homomorphism between the classes of elements of $\mathbf{W}$ and $\mathbf{DT}$ groups, and the $\text{SU}(3)$-conjugacy classes of elements of $\mathbf{DT}$.

The character values afforded by the three conjugacy classes of $\mathbf{W}$ are easily deduced using the action of representative elements on the points of a generic orbit $(a,b)$, illustrated on Fig. 1.

The decomposition of Weyl group orbits on the $A_2$ weight lattice into direct sums of irreducible representations of $\mathbf{W}$ is presented in Table VI.

The structure of the Demazure–Tits subgroup $\mathbf{DT}$ of $\text{SU}(3)$ is found either from the $\text{SU}(n)$ case of Sec. II or by a direct computation. It turns out to be the octahedral

$$\{p, q \in \mathbb{Z}, (2p + q \geq 2a + b, p + 2q \geq a + 2b) \}.$$  

(4.8)
**TABLE VI.** Decomposition of the orbits of the Weyl group acting as a permutation group on the \( A_1 \) lattice. Character of each class on the orbits is shown.

<table>
<thead>
<tr>
<th>( W ) orbit</th>
<th>Shape</th>
<th>( E )</th>
<th>( C_2 )</th>
<th>( C_1 )</th>
<th>( W ) orbit decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>((ab))</td>
<td>hexagonal</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>( \Gamma_1 \oplus \Gamma_2 \oplus 2\Gamma_3 )</td>
</tr>
<tr>
<td>( a,b &gt; 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((0))</td>
<td>triangular</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>( \Gamma_1 \oplus \Gamma_2 )</td>
</tr>
<tr>
<td>( a,b &gt; 0 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((0))</td>
<td>point</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>( \Gamma_1 )</td>
</tr>
</tbody>
</table>

The character table is in Table V. Each element of \( W \) corresponds to four elements of \( DT \). The correspondences are shown in Table V. The irreducible representations \( \Gamma_1, \Gamma_2, \) and \( \Gamma_ 3 \) of \( DT \) coincide with \( \Gamma_1, \Gamma_2, \Gamma_3 \) of \( W \). Our notation \( \Gamma_1, i = 1, ..., 5 \), for the representations of the octahedral group are taken from Ref. 16. Table V contains as well a sample element of each conjugacy class of \( DT \) and \( W \), and its \( SU(3) \) conjugacy class is identified in the case of \( DT \).

Table VI contains the decomposition of \( W \) orbits in the weight lattice \( Q \) into direct sums of irreducible components. Let us point out that the action of \( W \) is reducible under a general linear transformation but cannot be further reduced when it is confined to permutations of the lattice points.

We now consider the decomposition of the \( W \) orbits into direct sums of irreducible representations of \( DT \). The results are summarized in Table VII.

The analysis is simplest for the generic (hexagonal) orbit; we need to consider only the classes \( C_1 \) and \( C_2 \) that correspond to Weyl class \( C_1 \). We use \( R_1 \) as the representative element for \( C_1 \). Its eigenvalue is \((-1)^{mn}\), where \( m_1 \) is the \( SU(2) \) weight in the \( \alpha_1 \) (horizontal) direction; thus the eigenvalue is \((-1)^{s_1}((-1)^{n},(-1)^{y}+b)\) each for \( 2n \) states of the orbit and the trace (character) for \( C_1 \) is \( 6n \) for \( a, b \) both even, \(-2n \) otherwise, as given in Table VII.

We can treat the two types of triangular orbit simultaneously by letting \( (b) \) stand for \((0,b)\) or \((b,0)\) according as \( b \) is positive or negative. Then \( b \) is the second weight component of the states of the orbit for which \( m_1 = 0 \). The classes \( C_1 \) and \( C_2 \) are treated as for the hexagonal orbit and have the characters given in Table VII. We must consider in addition the classes \( C_4 \) and \( C_4' \) whose representatives we take as \( R_4 \) and \( R_4'^2 \), respectively. Only the \( m_1 = 0 \) states contribute to their trace; for them the eigenvalue of \( R_2 \) is \((-1)^{y}\) and that of \( R_1 \) is \((-1)^{s_1/2}\), where \( s_1 \) is the representation label of the \( SU(2) \) group in the \( \alpha_1 \) direction (\( s_1 \) is even for such states).

We will now derive a generating function for the characters of the classes \( C_4 \) and \( C_4' \). The generating function for \( SU(3) \supset SU(2) \times U(1) \) is

\[
F(P,Q,S,Z) = \frac{((1 - P)Z_1)(1 - PZ_2)}{1 - QSZ + (1 - QZ_2)^{-1}}.
\]

(4.9)

In the expansion of (4.9) the coefficient of \( P^r Q^s S^t Z^u \) is the multiplicity of the irreducible representation \( (s,t,z) \) of \( SU(2) \times U(1) \) in \( (p,q) \) of \( SU(3) \). To convert (4.9) to a generating function for the \( C_4 \) characters we retain only the part even in \( S \) [only even \( s \) representations of \( SU(2) \) contain an \( m = 0 \) state], set \( S^2 = -1 \) [the eigenvalue of \( R_1 \) is \((-1)^{s_1/2}\)], set \( Z = \sqrt{B} \), and separate the result into non-negative and negative powers of \( B \). The non-negative power part turns out to be

**TABLE VII.** Decomposition of orbits of the Demazure–Tits group in an \( SU(3) \) representation \((p,q)\) into the direct sum of irreducible representations \( \Gamma_1, ..., \Gamma_5 \) of \( DT \). A \( DT \) orbit is specified by an \( SU(3) \) dominant weight \((a,b)\); \( n \) is the multiplicity of \((a,b)\) in \((p,q)\). It is known that for \((0,0)\) weight \( n = 1 + \min(p,q) \); \( k = p - q \mod 2 \).

<table>
<thead>
<tr>
<th>DT orbit in ((p,q))</th>
<th>Decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dominant weight</strong></td>
<td><strong>Characters</strong></td>
</tr>
</tbody>
</table>
| \((a,b)\) \( a,b > 0 \) | \( \begin{array}{l}
6n \\
6n - 2n \\
3n \\
3n - n \\
n \\
n \\
0 \\
0 \\
n \\
\end{array} \) | \( \begin{array}{l}
n \\
n/2 \\
(n + 1)/2 \\
(n - 1)/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
\end{array} \) | \( \begin{array}{l}
n \\
n/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
(n + 1)/2 \\
\end{array} \) | \( \begin{array}{l}
ab \text{ even} \\
b \text{ even} \\
b \text{ odd}, n \text{ even} \\
b \text{ even}, n \text{ odd}, p - q \text{ even} \\
b \text{ even}, n \text{ odd}, p - q \text{ odd} \\
b \text{ odd}, p - q \text{ odd} \\
b \text{ odd}, p - q \text{ even} \\
\end{array} \) |
The coefficient of $P^3Q^4B^b$ in the expansion of (4.10) is the character of the class $C_4$ in the orbit $(0, 0)$ in $(p, q)$ of SU(3). The three terms in (4.10) never overlap (at most one contributes to the character in each case) and the character is $(-1)^{p+q+b}$ for $n$ odd, 0 for even, as shown in Table VII.

To get the $C_1$ character, replace $B$ by $-B$ in the generating function, or equivalently, multiply the $C_4$ character by $(-1)^b$. The characters for $(-b, 0)$ orbits are obtained from the negative power $(in B)$ part of the generating function with similar results, found in Table VII.

Finally, we come to the $(0, 0)$ point orbit. The characters of $C_1$, $C_2$, $C_3$, $C_4$ are found as before. In addition we now get nonzero contributions from $C_5$. Since $C_5$ contributes nothing to the characters of other orbits, its character for the point orbit is equal to that for the whole irreducible representation of SU(3). It is given by the generating function:

\[
(1 - PQ)/(1 - P^3)(1 - Q^3),
\]

i.e., 1 for $p = q = 0$ mod 3, $-1$ for $p = q = 1$ mod 3, 0 for $p = q = 2$ mod 3, as shown in Table VII. There is no point orbit for $p - q \neq 0$ mod 3.

V. THE DEMAZURE–TITS SUBGROUP OF $B_2$

The irreducible representation $(p, q)$ of the Lie algebra $B_2$ [or Lie group Sp(4) and also O(5)] has the highest weight $p\omega_1 + q\omega_2$; in particular, (1, 0) and (0, 1) are the representations of dimensions 5 and 4, respectively. Similarly $(a, b)$, $a, b > 0$, denotes a dominant weight or the Weyl group orbit of the $B_2$ lattice containing $(a, b)$; the multiplicity of $(a, b)$ in the weight system of $(p, q)$ is denoted by $n$.

The multiplicity $n$ of a dominant weight $\Lambda^+ = (a, b)$ is the coefficient of the term $P^3Q^4A^2B^b$ in the power expansion of the generating function

\[
\frac{1}{(1 - P)(1 - PA)(1 - Q^3)(1 - QB)} \times \left( \frac{1 + PQB}{(1 - P^3B^2)(1 - P^2)} + \frac{Q^2}{(1 - P)(1 - Q^2)} \right). 
\]

### TABLE VIII.

The character table of the groups DT($B_2$) and W($B_2$). Subscripts of the class symbol indicate the order of its elements. Here EFO denotes a $B_2$-conjugacy class; IR is an irreducible representation.

<table>
<thead>
<tr>
<th>Class</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>1</th>
<th>2</th>
<th>Rep.</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>C_1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C_2</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C_3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>C_4</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>C_5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

| Number of elements | 1 | 2 | 4 | 4 | 2 | 2 | 2 | 2 | 1 | 2 | 4 | 4 |

| EFO | [000] | [010] | [001] | [011] | [100] | [101] | [110] | [111] | [010] | [001] | [011] | [100] | [101] | [110] | [111] |

### DEMAZURE–TITS GROUP

Michel, Patera, and Sharp
The character tables of the \( W \) and \( DT \) groups are given in Table VIII. The character values of the five conjugacy classes of \( W \) are found from the action of representative elements on the points of a generic orbit \((a,b)\), \(a>0, b>0\), illustrated in Fig. 2. Thus one finds the decomposition of the Weyl orbits into the direct sums shown in Table IX.

We turn to the decomposition of \( DT \) orbits of an arbitrary irreducible representation \((p,q)\) of \( B_2 \). As usual the analysis is simplest for the generic (octagonal) orbit \((a,b)\) with \(a>0\) and \(b>0\); only the classes \(C_1, C_2, C_3\), which correspond to \( W \) class \( C_1 \), have nonzero characters. The weight vectors are eigenvectors of these classes' representative elements with the following eigenvalues:

\[
I \rightarrow 1, \quad R_1^2 \rightarrow ( -1 )^m, \quad R_2^2 \rightarrow ( -1 )^m.
\]  

\[
F(p,q;S_2;Z) = \frac{1}{(1 - PZ^2)(1 - PZ^{-2})(1 - QS_2Z)(1 - QS_2Z^{-1})} \left( \frac{1}{1 - PS_2^2} + \frac{Q^2}{1 - Q^2} \right).
\]  

(5.2)

In the expansion of (5.2) the coefficient of \( P^q Q^s S_2^b Z^a \) is the multiplicity of the representation \((s_2,z)\) of \( SU(2) \times U(1) \) in \((pq,q)\) of \( Sp(4) \). To convert (5.2) to a generating function for half the \( C_4 \) character (because two states contribute), we retain the part even in \( S_2 \) [only odd-dimensional \( SU(2) \) representations have even valued weights, in particular, the weight \( m_2 = 0 \)]. Then we set \( S_2^2 = -1 \) [the eigenvalue of \( R_2 \) is \(( -1 )^{5/2} \)], and set \( Z^2 = A \) and keep only the positive power part in \( A \). The result is

\[
\frac{1}{(1 - P^2)(1 + P)(1 + Q^2 A)} \left( \frac{1}{1 - PA} + \frac{Q^4 - PQ^3}{1 - Q^4} \right).
\]  

(5.3)

Twice the coefficient of \( P^q A^a \) is the character of \( C_4 \) for the orbit \((a,0)\). To get a generating function for half the \( C_2^* \) character substitute \( A \rightarrow - A \) in (5.3) or, equivalently, multiply the \( C_4 \) character by \(( -1 )^{a} \). The coefficients of the expansions have been evaluated and the results are summarized in Table IX. We give below the multiplicity \( n \) of \((a,0)\) orbits, obtained from the generating function (5.1) with \( B = 0 \), for all six cases \( q \) is even and \( p + \frac{1}{2} q \) is even:

\[
\begin{align*}
(1) & \quad p, \quad \frac{1}{2} q \geq a, \quad p - a \text{ even}, \quad n = 1 + \left( \frac{1}{2} pq + p + q - a^2 \right), \\
(2) & \quad p, \quad \frac{1}{2} q \geq a, \quad p - a \text{ odd}, \quad n = \left( \frac{1}{2} pq + p + q - a^2 + 1 \right), \\
(3) & \quad p \geq a \geq \frac{1}{2} q, \quad p - a \text{ even}, \quad n = \left( \frac{1}{2} pq + q + 3 \right) + \left( \frac{1}{2} (p - a)(q + 1) + 1 \right), \\
(4) & \quad p \geq a \geq \frac{1}{2} q, \quad p - a \text{ odd}, \quad n = \left( \frac{1}{2} pq + q + 3 \right) + \left( \frac{1}{2} (p - a)(q + 1) + 1 \right), \\
(5) & \quad \frac{1}{2} q \geq a \geq p, \quad n = \left( \frac{1}{2} (p + 1)(p + q - 2a + 2) \right), \\
(6) & \quad a \geq \frac{1}{2} q \geq p, \quad n = \left( \frac{1}{2} (p + \frac{1}{2} q - a + 1)(p + \frac{1}{2} q - a + 2) \right).
\end{align*}
\]  

(5.4)

\[
\begin{array}{cccccc}
\text{W orbit} & \text{Shape} & \text{Characters} & \text{W orbit decomposition} \\
(a,b), a,b > 0 & \text{octagon} & \begin{array}{cccc}
C_1 & C_2 & C_3 & C_7 \\
8 & 0 & 0 & 0 \\
4 & 2 & 0 & 0 \\
4 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 \\
\end{array} & \Gamma \rightarrow \Gamma \vee \Gamma \vee \Gamma \vee 2 \Gamma \vee \Gamma \\
(a,0), a > 0 & \text{square} & \begin{array}{cccc}
C_1 & C_2 & C_3 & C_7 \\
8 & 0 & 0 & 0 \\
4 & 2 & 0 & 0 \\
4 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 \\
\end{array} & \Gamma + \Gamma + \Gamma \\
(0,b), b > 0 & \text{square} & \begin{array}{cccc}
C_1 & C_2 & C_3 & C_7 \\
8 & 0 & 0 & 0 \\
4 & 2 & 0 & 0 \\
4 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 \\
\end{array} & \Gamma + \Gamma + \Gamma \\
(0,0) & \text{point} & \begin{array}{cccc}
C_1 & C_2 & C_3 & C_7 \\
8 & 0 & 0 & 0 \\
4 & 2 & 0 & 0 \\
4 & 0 & 2 & 0 \\
1 & 1 & 1 & 1 \\
\end{array} & \Gamma \vee \Gamma \vee \Gamma \vee 2 \Gamma \vee \Gamma \\
\end{array}
\]  

Table IX. Decomposition of the orbits of the Weyl group \( W(B_2) \) acting as a permutation group on the \( B_2 \) lattice. Characters of each class on the orbit are shown.


Michel, Patera, and Sharp
For the square orbit \((0, b)\), with diagonal sides, the classes with nonzero trace are \(C_1, C_2, C_3^\prime, C_4^\prime, C_5^\prime, C_6^\prime, C_7^\prime,\) and \(C_8^\prime\). The characters of \(C_1, C_2, C_3^\prime\) are found as for the octagonal \(C_4\) orbit. We take the representative elements of \(C_1, C_2, C_3^\prime, C_4^\prime, C_5^\prime, C_6^\prime, C_7^\prime,\) and \(C_8^\prime\) to be, respectively, \(R_1, R_1^3, R_2, R_2^3, R_3^2R_1,\) and \(R_3^2R_1^3\). Only the top and bottom \((m_1 = 0)\) states of the orbit contribute to their characters; the eigenvalue of \(R_i\) is \((-1)^i\) and that of \(R_3^2\) is \((-1)^i\) for these states. We now derive a generating function for the characters of the classes in question.

The generating function for \(\text{Sp}(4) \supset \text{SU}(2) \times \text{SU}(2)\) branching rules is

\[
F(P, Q; S_1, U) = [(1 - P)(1 - PS_1U)(1 - QS_1)(1 - QU)]^{-1}.
\]

(5.5)

In the expansion of (5.5) the coefficient of \(P^aQ^bS_1^cU^d\) is the multiplicity of the representation \((s_1, u)\) of \(\text{SU}(2) \times \text{SU}(2)\) in \((p,q)\) of \(\text{Sp}(4)\); here \(s_1\) is the \(\text{SU}(2)\) representation label (highest weight) in the direction of \(\alpha_1\) and \(u\) is the representation label in the \(\alpha_1 + 2\alpha_2\) (vertical) direction. To convert (5.5) into a generating function for half (because two states contribute) the \(C_1^\prime\) character, we retain the part of (5.5) that is even in \(S_1\) [only even \(s_1\) representations of \(\text{SU}(2)\) have states with \(m_1 = 0\)]. Set \(S_1^2 = -1\) [the eigenvalue of \(R_3^2\) is \((-1)^i\)], multiply by \((1 - U^{-2})(1 - U^{-2}B)\) and keep the \(U^0\) part (thereby retaining only positive \(u\) weights, which are just the orbit labels). The result is

\[
\frac{1}{(1 + P^2)(1 + Q^2)}\left[\frac{1}{(1 - P)(1 + P^2Q^2)}\right]
\]

(5.6)

Twice the coefficient of \(P^aQ^bS_1^cU^d\) is the character of \(C_4^\prime\) for the orbit \((0, b)\). To get a generating function for half the characters of \(C_{4e}^\prime\) (and \(C_{4o}^\prime\)) for the orbit \((0, b)\), substitute \(B = -B\) in (5.6) or, equivalently, multiply the \(C_4^\prime\) characters by \((-1)^i\). The coefficients have been evaluated (they take only the values \(\pm 1\) and 0) and the result is found in Table XII, along with the reduction of \((0, b)\) to the direct sum of irreducible representations of \(\text{DT}\). We give below the multiplicity \(n\) for \((0, b)\) orbits, obtained from the generating function (5.1) with \(A = 0\). For each case \(q - b\) is even and \(p + q\) is odd.

\[
\begin{align*}
\text{Case} & \quad \text{Value} \\
(1) & \quad q > b; \\
(2) & \quad p \text{ odd}; \quad q > b; \\
(3) & \quad p \text{ even}; \quad q < b; \\
(4) & \quad p \text{ odd}; \quad q < b;
\end{align*}
\]

TABLE XI. Decomposition of square orbit \((a0)\) of \(\text{DT}(B_3)\) into the direct sum of its irreducible representations. Only nonzero characters are shown. The values of the multiplicity \(n\) are given in (5.4); \(\alpha = (-1)^{(p + |q| + a + 2)}\), \(\beta = (-1)^{(p + |q| - a + 1)}\), \(\gamma = p + 2\), \(\delta = p + 1\).
TABLE XII. Decomposition of square orbit \((0, b)\) of \(DT(B_2)\) into irreducible representations of \(DT(B_2)\). Characters not shown are 0. Values of the multiplicity \(n\) are given in (5.7). \(p \equiv b \mod 4\), we have

\[\begin{align*}
\alpha &= +1, \text{ for } \left(p \mod 4, q \mod 4, b \mod 4\right) = (0,0,0), (0,1,1), (0,1,3), (0,2,2), (1,0,0), (2,2,2); \\
\alpha &= -1, \text{ for } \left(p \mod 4, q \mod 4, b \mod 4\right) = (1,2,0), (2,0,2), (2,1,1), (2,1,3), (2,2,0), (3,0,2); \\
\alpha &= 0, \text{ otherwise.}
\end{align*}\]

For \(p \equiv b \mod 4\), we have

\[\begin{align*}
\alpha &= +1, \text{ for } \left(p \mod 4, q \mod 4, b \mod 4\right) = (0,0,0), (0,1,1), (0,2,2), (0,3,3); \\
\alpha &= -1, \text{ for } \left(p \mod 4, q \mod 4, b \mod 4\right) = (2,2,0), (2,3,1), (2,0,2), (2,1,3); \\
\alpha &= 0, \text{ otherwise.}
\end{align*}\]

<table>
<thead>
<tr>
<th>Characters</th>
<th>Decomposition</th>
<th>Restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_1)</td>
<td>(\gamma)</td>
<td>((n + \alpha) (\Gamma_1 \oplus \Gamma_3) + \frac{1}{2} (n - \alpha) (\Gamma_1 \oplus \Gamma_3) + n \Gamma_5)</td>
</tr>
<tr>
<td>(C_2)</td>
<td>(\gamma)</td>
<td>((n + \alpha) (\Gamma_1 \oplus \Gamma_3) + \frac{1}{2} (n - \alpha) (\Gamma_1 \oplus \Gamma_3) + n \Gamma_5)</td>
</tr>
<tr>
<td>(C_i)</td>
<td>((n + \alpha) (\Gamma_1 \oplus \Gamma_3) + \frac{1}{2} (n - \alpha) (\Gamma_1 \oplus \Gamma_3) + n \Gamma_5)</td>
<td>(-1) if (p \equiv 0 \mod 4), (q \equiv 0 \mod 4); (-1) if (p \equiv 1 \mod 4), (q \equiv 0 \mod 4); (-1) if (p \equiv 1 \mod 4), (q \equiv 2 \mod 4); (-1) if (p \equiv 2 \mod 4), (q \equiv 2 \mod 4); 0, otherwise.</td>
</tr>
</tbody>
</table>

In the above

\[\begin{align*}
\beta &= \min \left\lfloor \frac{b}{2}, \frac{p}{2} \right\rfloor, \\
\gamma &= \min \left\lfloor \frac{b + 1}{2}, \frac{p}{2} \right\rfloor, \\
\delta &= \min \left\lfloor \frac{b - 1}{2}, \frac{p}{2} \right\rfloor, \\
\epsilon &= \min \left\lfloor \frac{b - 1}{2}, \frac{p - 1}{2} \right\rfloor,
\end{align*}\]

Finally we turn to the \((0,0)\) point orbit. The characters \(C_i\) are found in the same way. We then obtain

\[n = \frac{1}{2} \left( \left( p - \delta - \xi + 1 \right) \left( \delta - \xi + 1 \right) + \left( p - \epsilon - \xi + 1 \right)\left( \epsilon - \xi + 1 \right) \right)\]  \(\text{(5.7)}\)

VI. THE DEMAZEUR-TITS SUBGROUP OF \(G_2\)

As in the previous two cases, \((p,q) = p\omega_1 + q\omega_2\) is the highest dominant weight denoting an irreducible representation of \(G_2\). In particular, \((1,0)\) and \((0,1)\) are the representations of dimensions 14 and 7, respectively. A dominant weight \((a,b) = a\omega_1 + b\omega_2\) denotes the \(W\) orbit in the \(G_2\)-weight (and also root) lattice containing it, as well as the \(DT\) orbit of subspaces in the representation space labeled by the highest weight \((p,q)\). Naturally one assumes that \((a,b) \in \Omega(p,q)\), otherwise our problem is trivial.

TABLE XIII. Decomposition of the point orbit of \(DT(B_2)\) into its irreducible representations. The values of \(n\) are given in (5.9). \(\alpha = (-1)^{p+2} \times \left(\frac{1}{2}(p + q) + 1\right), \beta = (-1)^{q+2} \left(\frac{1}{2}(p + 1)\right), \gamma = 2(-1)^{q+2}, \delta = 2(-1)^{q-2}\).

<table>
<thead>
<tr>
<th>Nonzero multiplicities of irreducible (DT(B_2)) representations</th>
<th>((p,q)) mod 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>((n + p + \alpha + \gamma + 4))</td>
<td>((n - p - a - \gamma))</td>
</tr>
<tr>
<td>((n + p + \alpha))</td>
<td>((n - p - a))</td>
</tr>
<tr>
<td>((n + p + b - \gamma + 2))</td>
<td>((n + p + b - \gamma + 3))</td>
</tr>
<tr>
<td>((n + p + b + \gamma - 1))</td>
<td>((n + p + b + \gamma - 2))</td>
</tr>
<tr>
<td>((n + p + b + \alpha + 2))</td>
<td>((n + p + a + 2))</td>
</tr>
<tr>
<td>((n + p + b - \alpha - 2))</td>
<td>((n + p - a - 2))</td>
</tr>
<tr>
<td>((n - p + b - 1))</td>
<td>((n - p + a - 1))</td>
</tr>
</tbody>
</table>
TABLE XIV. Character table of the DT(G2) and W(G2) groups. Representative element of each conjugacy class is shown. Subscript on class symbol is the order of its elements. Conjugacy classes of G2 are given as E10. IR is an irreducible representation.

<table>
<thead>
<tr>
<th>Class</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$r_5$</th>
<th>$r_6$</th>
<th>Representative element</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$I$</td>
<td>1</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$C_2$</td>
<td>$C_3$</td>
<td>$C_4$</td>
<td>$C_5$</td>
<td>$C_6$</td>
<td>$C_7$</td>
<td>$C_8$</td>
<td>2</td>
</tr>
<tr>
<td>$r_1$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$r_1$</td>
<td>1</td>
</tr>
<tr>
<td>$r_2$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$r_2$</td>
<td>1</td>
</tr>
<tr>
<td>$r_3$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$r_3$</td>
<td>1</td>
</tr>
<tr>
<td>$r_4$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$r_4$</td>
<td>1</td>
</tr>
<tr>
<td>$r_5$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$r_5$</td>
<td>1</td>
</tr>
<tr>
<td>$r_6$</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>$r_6$</td>
<td>1</td>
</tr>
</tbody>
</table>

The multiplicity $n = \text{mult}_{(p,q)} (a,b)$ of a weight $(a,b)$ in the weight system $\Omega(p,q)$ is also the multiplicity of the DT orbit. It can be found either in the tables of Ref. 13 (for the lowest 100 representations) or it can be calculated using the $G_2$ character generator, Eq. (2.7) of Ref. 18. There in order to conform to present notation the following substitutions should be made: $A \rightarrow Q, B \rightarrow P, \eta \rightarrow AB^{-3/2}, \xi \rightarrow B^{1/2}$, then the coefficient of the term $P^C \partial^C A^b B^b (a,b \text{ non-negative})$ in the power expansion of the generating function is the multiplicity $n$.

The character table of the Weyl group $W(G_2)$ and the Demazure–Tits group DT$(G_2)$ are found in Table XIV.

First consider W acting on the $G_2$ weight lattice. Representative elements of the W-conjugacy classes are

$$C_1: I, \quad C_2: (r_1 r_2)^3, \quad C_3: r_2, \quad C_4: r_1, \quad C_5: (r_1 r_2)^3, \quad C_6: r_2 r_1 r_2.$$  \hspace{1cm} (6.1)

The subscript on a class symbol is the order of its elements; $r_1$ and $r_2$ are the elementary reflections (2.1). The traces of classes of each type are easy to determine as before: each point of the orbit that is not moved by the representative element contributes 1 to the trace. Hence it suffices to see the action of the representative of each class on $Q(G_2)$. It is shown in Fig. 3.

Consider the generic, or dodecagonal, orbit $(a,b)$, $a > 0$, $b > 0$, of the Weyl group in the $G_2$ weight lattice $Q$. The class $C_1$ has trace 12, while all other classes have trace 0. Hence one has the decomposition $(a,b) = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$, as shown in Table XV. Similarly for the hexagonal orbit $(a,0)$, $a > 0$, the class $C_1$ has trace 6, the class $C_2$ has trace 2, and all other classes have trace 0. We find the decomposition $(a,0) = \Gamma_1 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$, (cf. Table XV).

For the other hexagonal orbit, $(0,b)$, $b > 0$, the class $C_1$ has trace 6, the class $C_2$ has trace 2, and the others are 0. The decomposition is $(0,b) = \Gamma_1 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$. Finally for the point orbit $(0,0)$ each class has trace 1 so that its decomposition is $(0,0) = \Gamma_1$. The decomposition of Weyl group orbits of $Q(G_2)$ is summarized in Table XV.

Next let us consider the DT group acting on the weight vector basis of $V_\Lambda$, $\Lambda = (p,q)$ and let us find the decomposition (3.11).

We consider first the generic orbit $(a,b)$, $a > 0$, $b > 0$, which appears with multiplicity $n$ in $V_{(p,q)}$. The classes with nonzero traces are $C_1$ and $C_2$. The trace of $C_1$ is 12n. For $C_2$ we have the representative element $R_2$; its eigenvalue is $(-1)^m$, where $m$ is the SU(2) weight in the $a_1$ direction. The values of $|m|$ at the 12 points of the orbit are $a, a + b$.

TABLE XV. Decomposition of the Weyl group orbits of the $G_2$ lattice.

<table>
<thead>
<tr>
<th>W orbit on $G_2$ lattice</th>
<th>Shape</th>
<th>Characters of classes</th>
<th>Orbit decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a,b)$</td>
<td>dodecagonal</td>
<td>$C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_6$</td>
<td>$\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$</td>
</tr>
<tr>
<td>$a,b &gt; 0$</td>
<td></td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>$(a,0)$</td>
<td>hexagonal</td>
<td>$C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_6$</td>
<td>$\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td></td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$(0,b)$</td>
<td>hexagonal</td>
<td>$C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_6$</td>
<td>$\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$</td>
</tr>
<tr>
<td>$b &gt; 0$</td>
<td></td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>$(0,0)$</td>
<td>point</td>
<td>1</td>
<td>$\Gamma_1$</td>
</tr>
</tbody>
</table>


Michel, Patera, and Sharp
2a + b, each 4n times. Hence the trace for $C_3$ is 12n for a, b both even, and $-4n$ otherwise. Hence one has the decomposition as given in Table XVI.

The hexagonal orbit (a,0), $a, b > 0$, has two horizontal sides; the classes with nonzero character are $C_{1}, C_{2}, C_{3}, C_{4}, C_{5}$, as follows from Fig. 3. The trace of $C_{1}$ is 6n. For $C_{2}$, the trace is 6n if $a$ is even, and $-6n$ if $a$ is odd. We will derive generating functions for traces of $C_{1}, C_{2}$, and $C_{3}$. Orient the $SU(2) \times SU(2)$ subgroup of $G$, so that $\sigma_{2}$ points in the direction of the second SU(2) root. The states not moved by $R_{3}$ and $R_{4}$, the representative elements of $C_{4}$ and $C_{3}$, respectively, are those with dominant weight $(a,0)$ and opposite weight $(a,0)$. On these states the eigenvalue of $R_{3}$ is $(-1)^{a/2}$, and of $R_{4}$ is $(-1)^{a}$; $|m_{2}|$ takes the value $2a$, where $(a, m_{2})$ is the representation label and weight of the first SU(2) subgroup and $(a, m_{2})$ those of the second.

The even-even part of the $G \rightarrow SU(2) \times SU(2)$ branching rules generating function is found from Ref. 18, Eq. (3.1) (to conform to our present notations, the substitutions $A \rightarrow Q$ and $B \rightarrow P$ should be made):

$$F(P, Q, S^{2}, T^{2}) = \frac{1}{(1 - P)(1 - Q^{2})(1 - Q^{2}S^{2})(1 - Q^{2}S^{2}T^{2})} \left[ 1 + PQS^{2} + Q^{2}S^{2}T^{2} + P^{2}QS^{4}T^{4} + PQS^{2}T^{2} + (1 - P^{2})(1 - Q^{2}S^{2}) \left( 1 + PQS^{2} + Q^{2}S^{2}T^{2} + P^{2}QS^{4}T^{4} + PQS^{2}T^{2} \right) \right]^{10}.$$

(6.2)

Because $R_{2} = (-1)^{1/2}$, we set $T^{2} = -1$. The result is

$$F'(P, Q, S^{2}) = \frac{1}{(1 - P^{2})(1 - Q^{2})} \left[ \frac{1}{(1 - P^{2})(1 - Q^{2})} \right]^{10}.$$

(6.3)

Finally we convert this generating function for SU(2) representations to the corresponding one for non-negative SU(2) weights (or $G_{2}$ orbit labels, since $a = |m_{2}|$) by computing

$$G(P, Q, A) = \frac{F'(P, Q, S^{2})}{(1 - S^{2})(1 - S^{2}A)}.$$

(6.4)

### Table XVI

Decomposition of $G_{2}$ orbits of the Demazure-Tits group $DT$ in a representation $(p,q)$ into a direct sum of irreducible representations $\Gamma_{(m)}$ of DT. An orbit is given by a $G_{2}$ dominant weight $(a, b)$; $n$ is the multiplicity of $(a, b)$ in $(p, q)$. Notation: $c, d, e, f, g$ are the coefficients of the term $P^{c}Q^{d}A^{e}B^{f}$ in the power series of Eqs. (6.5), (6.8), (6.9), (6.10), (6.11), respectively; $X_{\pm} = (n \pm \epsilon)/12$, $Y_{\pm} = (d \pm \epsilon)/4$, $Z_{\pm} = (f \pm g)/6$.

| DT orbit in $(p,q)$ | Decomposition | Multiplicities of irreps of DT$|G_{2}|$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
<tr>
<td>$\Gamma_{(m)}$</td>
<td></td>
<td>$\Gamma_{1}$ $\Gamma_{2}$ $\Gamma_{3}$ $\Gamma_{4}$ $\Gamma_{5}$ $\Gamma_{6}$ $\Gamma_{7}$ $\Gamma_{8}$ $\Gamma_{9}$ $\Gamma_{10}$</td>
</tr>
</tbody>
</table>
The symbol \( |_S^n \) indicates that only the 0th power of \( S \) term of (6.4) should be retained. The power series expansion of \( G(P,Q,A) \),

\[
G(P,Q,A) = \sum_{pq} P^p Q^q A^r c_{pq},
\]

states that the trace of class \( C_i \) is \( 2c_{pq} \) for the orbit \((a,0)\) in \((p,q); \) for \( C_i \) the trace is \( 2(-1)^r c_{pq} \). The factor 2 appears because two states contribute to the trace. The decomposition of the \((a,0)\) orbit is shown in Table XVI.

For the hexagonal orbit \((0,b), b > 0 \) (two vertical sides), the classes with nonzero trace are \( C_i, C_j, C_k, C_l \). For \( C_i \) the trace is \( 6n \). For \( C_j \) it is \( 6n \) for \( b \) even, \(-2n \) for \( b \) odd. We derive generating functions for the trace of \( C_i \) and \( C_j \), using the representative elements \( R_jR_i \) and \( R_j \), respectively. Orient the \( SU(2) \times SU(2) \) subgroup with the first \( SU(2) \) root along \( \alpha_i \) of \( G_i \). The states not moved by \( R_i \) and \( R_jR_i \) are those with weights \((0, \pm b)\). On these states the eigenvalue of \( R_i \) is \((-1)^{\sigma_i} [5 \) is the first \( SU(2) \) representation label] and that of \( R_j \) is \((-1)^{\phi} \); \( m_i \) takes the value \( 2b \) \( m_i \) is the second \( SU(2) \) weight. Because \( R_j \) is \((-1)^{\sigma_j} \), we set \( S^2 = -1 \) in the generating function (6.2) with the result

\[
F''(P,Q,T^2) = \frac{1}{(1 - P^2)(1 + Q^2)} \left[ \frac{1}{1 - Q^2} \left( \frac{1}{1 + Q^3} + \frac{QT^2}{1 - Q^2} \right) \right] - \frac{P^2 \left( \frac{1}{1 - Q^2} \left( \frac{1}{1 + Q^3} + \frac{QT^2}{1 - Q^2} \right) \right)}{P - P^2 \frac{1}{1 + Q^3} + \frac{PQ^2}{1 - Q^2}}.
\]

Finally we convert this generating function for \( SU(2) \) representations into the corresponding one for non-negative weights (or \( G_i \) orbits labels, since \( b = m_i \)) by computing

\[
H(P,Q,B) = \frac{H(P,Q,T^2)}{1 - T^{-2}} (1 - T^{-2}B)
\]

\[
= \frac{1}{1 + Q^2} \left[ \frac{1 + Q^2}{1 - P^2} \frac{1}{1 + Q^3} + \frac{P^2}{1 + P} \frac{1}{1 + Q^3} \right] - \frac{PQ^2}{P - P^2} \frac{1}{1 + Q^3} + \frac{PQ^2}{1 - Q^2}
\]

\[
= \frac{P^2}{1 - P^2} \frac{1}{1 - Q^2} \left( \frac{1}{1 + Q^3} + \frac{PQ^2}{1 - Q^2} \right) + \frac{PQ^2}{1 - P^2} \frac{1}{1 + Q^3} + \frac{PQ^2}{1 - Q^2}
\]

\[
= \frac{P^2}{1 - P^2} \frac{1}{1 - Q^2} \left( \frac{1}{1 + Q^3} + \frac{PQ^2}{1 - Q^2} \right) + \frac{PQ^2}{1 - P^2} \frac{1}{1 + Q^3} + \frac{PQ^2}{1 - Q^2}
\]

The power series expansion of \( H(P,Q,B) \),

\[
H(P,Q,B) = \sum_{pq} P^p Q^q B^d c_{pq}, \tag{6.8}
\]

gives the trace of the class \( C_i \) for the orbit \((0,b)\) in the \( G_i \) representation \((p,q)\) as \( 2d_{pq} \); for \( C_k \) the trace is \( 2(-1)^{\phi} c_{pq} \). The decomposition of the orbit \((0,b)\) is given in Table XVI.

Finally we deal with the point orbit \((0,0)\). All classes can now have nonzero trace. The traces of classes \( C_i, C_j, C_k, C_l, C_m, C_n \) are computed as above for the hexagonal orbits. Thus the trace of \( C_i \) and \( C_j \) is \( n \), the multiplicity of the orbit. A generating function for \( n \) is obtained from (6.2) by setting \( S = T = 1 \), since each even \( SU(2) \times SU(2) \) representation has just one state at the origin; \( n \) for \((p,q)\) is the coefficient of \( P^r Q^s \) in the power series expansion. For \( C_i \), \( n \) the trace is \( c = c_{pq} \), the coefficient of \( P^r Q^s A^t \) in the expansion of (6.4). For \( C_j, C_k \), the trace \( d = d_{pq} \), the coefficient of \( P^r Q^s B^t \) in the expansion of (6.7). Since the remaining classes have zero trace for all but the point orbit, their trace for the point orbit is their character in the whole irreducible representation \((p,q)\). Accordingly we can get it from the known generating functions for the characters of the corresponding \( G_i \)-conjugacy class of elements of finite order in \( G_i \), Ref. 17. For \( C_i \) and \( C_j \) we have

\[
\sum_{pq} P^p Q^q c_{pq} = \frac{1}{(1 + P)^2(1 - P^2)^2(1 + Q^2)(1 - Q^2)^2} \times [1 + P - 2PQ - P^3Q - PQ^2 + P^2Q^2 - 2PQ^3 + P^3Q^3 - P^2Q^4 + \ldots]
\]

\[
= \frac{1}{(1 - Q^2)(1 - P - P^3Q - P^2Q^2 - 2P^2Q^2 - P^3Q^3) + P^2Q^4 + P^3Q^4 \ldots}
\]

\[
\sum_{pq} P^p Q^q c_{pq} = \frac{1}{(1 + P)^2(1 + Q^2)(1 + Q^2)} \times [1 + P + 2Q + 2Q^2 + PQ^2 + PQ + Q^3 + P^2Q^2 + P^4Q^2 + \ldots]
\]

\[
= \frac{1}{(1 - Q^2)(1 - P - P^3Q - P^2Q^2 - 2P^2Q^2 - P^3Q^3) + P^2Q^4 + P^3Q^4 \ldots}
\]
Our result, the decomposition of the point orbit, is given in Table XVI.

VII. CONJUGACY CLASSES OF ELEMENTS
GENERATING THE DEMAZURE–TITTS GROUPS

In this section we consider the elements \( R_k, \ k \in \{1,2,\ldots, 7\} \), which generate the Demazure–Tits group \( DT(G) \) up to equivalence transformation by the simple connected Lie group \( G \), and identify the \( G \)-conjugacy classes to which they belong. Since part of that has been done already in Ref. 7, here we just complete Table III of that article.

First let us show that \( R_k, \ k \in \{1,2,\ldots, 7\} \), are rational elements in any \( G \). (An element is rational if its character values for any representation of \( G \) are integers.) Consider \( R_k \in SU_k(2) \subseteq G \), and the subgroup \( SU_k(2) \) whose simple root is \( \alpha_k \). The character value of \( R_k \) for any representation \( \Lambda(G) \) of \( G \) is by definition its character for the subgroup representation \( \Lambda(SU_k(2)) \subseteq \Lambda(G) \). Then recalling the fact that \( R_k \) is a rational element of \( SU_k(2) \), it has to be rational also in \( G \).

We know that all \( R_k \) are of order 4 and that those \( R_k \) corresponding to simple roots \( \alpha_k \) of the same length are \( G \)-conjugate, while any two \( R_k \) corresponding to roots of different lengths are not \( G \)-conjugate. Therefore here we have to identify one conjugacy class of elements of order 4 in \( D_4 \), \( E_6 \), \( E_7 \), and \( E_8 \) and two such conjugacy classes in \( F_4 \). For all other cases the conjugacy classes were found. All the conjugacy classes of \( R_k \) are shown in Table XVII.

From now on we assume the conventions and results of Ref. 7. In particular, elements of finite order in \( G \) are denoted by relatively prime non-negative integers attached to the nodes of extended Coxeter–Dynkin diagram; we use the Dynkin numbering of the nodes (cf., for instance, Ref. 7 or Ref. 13). It is not difficult to list all conjugacy classes of elements of order 4 in any \( G \). Thus, for example, there are only seven such classes of elements in \( E_8 \). Since this is clearly the most complicated case we have to face, we illustrate in this example how one can proceed.

Let \( g \in E_8 \) belong to one of the seven \( E_8 \)-conjugacy classes of elements of order 4, \( g^4 = 1 \). Note that all \( E_8 \) representations are self-contragredient. Therefore \( g \) and \( g^{-1} = g^3 \) are conjugate, \( g = g^3 \). That is, all powers of \( g \) relatively prime to 4 are conjugate to \( g \). Consequently, the character \( \chi_\Lambda(g) \) of any element of our seven conjugacy classes is an integer in any representation \( \Lambda \) of \( E_8 \).

Since all eight \( R_k, \ k \in \{1,2,\ldots, 8\} \), are \( E_8 \)-conjugate, it suffices to consider only, say, \( R_1 \). In adopted conventions the \( E_8 \) simple roots are numbered as

\[ \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7. \]

Let us find the character value \( \chi_{Ad(E_8)}(R_1) \) of \( R_1 \) on the 248-dimensional (adjoint) representation \( Ad(E_8) \). But \( h_1 \) is orthogonal to the diagram with \( \alpha_1 \) removed. That is, the diagram of \( E_7 \) (its dimension is 133) on which the SU(2) with simple root \( \alpha_1 \) acts trivially. Otherwise \( R_1 \) sends \( h_1 \) to \( -h_1 \) and merely transposes the remaining root vectors in pairs which contributes nothing to the character. Therefore one has

\[ \chi_{Ad(E_8)}(R_1) = 132, \ k \in \{1,2,\ldots, 8\}. \]

Next we find which of the seven elements of order 4 in \( E_8 \) has that character value on \( Ad(E_8) \). It turns out that there is just one such element \([21000000]\). Using the extended diagram, it is given as

\[ 0 \]

In order to verify that its character is indeed 132, one can consult the table of positive roots of \( E_8 \) (pp. 62 and 63 of Ref. 13), this time reading the roots in the simple root basis \( (\alpha \text{ basis}) \). We need to know only the \( \alpha_1 \) coordinate of each root. That coordinate takes only five values \( 0, 1, 2, 3, 4 \), negative values occurring for negative roots only. An \( E_8 \) root with the \( \alpha_1 \) coordinate \( m \) contributes to the character value \( \exp(2\pi im/4) \). Moreover since the character must be integer, the values \( m = 1 \) and \( 3 \) can be disregarded; they must cancel out. Among the positive roots one finds 63 times \( m = 0 \) and once \( m = 2 \); the negative roots contribute similarly. Adding the eight zero weights of the adjoint representation as another \( m = 0 \) eight times, one gets the character as 132. In the same way, but much more quickly, one can determine the rest of the conjugacy classes of \( R_k \) in any other simple \( G \).

VIII. CONCLUDING REMARKS

The Weyl group has been the most important device in virtually any extensive work with representations of high rank (⩾ 1) simple Lie algebras/groups. The higher the rank the more difficult it is to proceed without it.

Physical states “live” in representation spaces rather than in spaces populated by roots of an algebra or weights of its representations. Consequently, the symmetries of the Weyl group are no more than an (homomorphic) image of the general symmetries of physical states. Moreover, interesting problems at any period of time are usually at (or beyond) the limits of what one can calculate with present day methods. Therefore using only the Weyl group is helpful but one can often proceed much more effectively.
A motivation to carry out large scale computations is often present in physics but only rarely in mathematics. That is perhaps the reason that a tool of prime importance like the Demazure–Tits group has been relatively little studied by mathematicians.

This independent sequel to Ref. 3 is an attempt to partially rectify the situation. The principal results are the following: Description of the DT in the classical series of simple Lie groups and $G_2$; identification of the conjugacy classes (under the Lie group action) of the elements generating $\text{DT}$; finding the character table of $\text{DT}$ in simple Lie groups of rank $2$; and decomposition of all finite-dimensional representations of rank $2$ Lie groups into direct sums of irreducible components of $\text{DT}$.

There remain unsolved other equally interesting problems involving $\text{DT}$. We name a few.

The character tables of $\text{DT}$ group in simple Lie groups of rank $> 2$. An extension of known character tables of $W$ to those of $\text{DT}$, as exemplified here for rank $2$, is possible and it may not even be difficult.

The structure of $\text{DT}$ in $E_6$, $E_7$, $E_8$, and $F_4$. The following appears to be true: $\text{DT}(E_6) \subset \text{DT}(E_{6+k})$ for $k = 6$ and $7$. The homomorphism $\text{DT}(E_6) \to W(E_6)$ is nonsplit.

Branching rules for Lie groups of rank $> 2$ to $\text{DT}$. The multiplicities of Weyl group orbits in corresponding weight systems are either known or can easily be found right now for every case which may conceivably ever be needed.

Integrity bases of invariants and covariants of $\text{DT}$. Their description along the lines, for instance, Ref. 16 is possible at least for lower ranks.

Let us finish the article with a remark concerning the action of $\text{DT}(G)$ on a generic orbit $V_W(\lambda^+)$ of its dominant weight $\lambda^+ = (\lambda_1, \ldots, \lambda_l)$ has only trivial stabilizer in $W$; equivalently, $\lambda^+$ has no zero coordinates in the basis of fundamental weights, $\lambda_j > 0$ for any $1 \leq j \leq l$. The decomposition (3.11) in this case depends only on the values $\lambda_j \mod 2$, $1 \leq j \leq l$ and not on the highest weight $\Lambda$ of any representation of $G$.

The only elements of $\text{DT}(G)$ which have nonzero trace on $V_W(\lambda^+)$ are the $2^l$ elements which are mapped under $\vartheta$ of (1.2) to the identity element of $W$. All other elements of $\text{DT}$ move every vector of $V_W(\lambda^+)$. The $2^l$ elements of the form

$$\prod_{i=1}^{l} \begin{pmatrix} R_i \end{pmatrix}^\delta_i, \, \delta_i = 0 \text{ or } 1.$$ 

The eigenvalue of $R_i$ acting on any vector of weight $\sum \lambda_i m_i \omega_i$ is just $(-1)^m$. The weight component $m_i$ is also the $SU(2)$ weight in the $\alpha_i$ direction.

The eigenvalues of all elements of $\text{DT}$ with nonzero trace thus depend only on the weights of the orbit. Their characters and hence their orbit decomposition, therefore depend only on the parity of $\lambda_j$'s.

**ACKNOWLEDGMENTS**

The authors are grateful for the hospitality of the Institute des Hautes Etudes Scientifiques (J. P. and R. S.), to the Centre de recherches mathématiques, Université de Montréal (L. M.), and to the Aspen Center for Physics (J. P.) where this work was pursued. We would also like to thank A. J. Coleman, R. Griess, and J. McKay for helpful remarks and for reading the manuscript.

**APPENDIX: A SUMMATION FORMULA**

Here we derive the following identity:

$$\sum_{x=0}^{(q/2)} (-1)^{x-y_q}(q-x)! = (q + 2) \mod 3 - 1,$$  \hspace{1cm} (A1)

which we have not been able to find in the literature. The right-hand side is the character of the conjugacy class $[111]$ of elements of finite order in $SU(3)$ on the irreducible representation $(p,q)$, $p \geq q$, $p - q = 0 \mod 3$, as given in Ref. 17, and used in Sec. IV of this paper. We may represent the EFO by $R_1 R_2$, an element of $\text{DT} \subset SU(3)$ belonging to the DT class $C_3$. Since it has trace zero on all but the point orbit, its trace for the point orbit is also given by the right-hand side of (A1). We show below that it is also given by the left-hand side of (A1).

The zero-weight space $V_{(p,q)}(0,0)$ is of dimension $q + 1$. It is spanned by the $q + 1$ vectors which can be written as

$$|x\rangle = (\eta \eta^*)^y \eta^\xi^z \xi \eta \eta^\xi^z (p-q/3), \, x = 0, 1, \ldots, q,$$  \hspace{1cm} (A2)

where $\eta, \xi, \xi^*$ are the three weight vectors of the $SU(3)$ representation $(1,0)$ of weights $(1,0)$, $(1,1), (0,1)$, respectively; $\eta \eta^* \xi, \xi^* \xi$ are the weight vectors of the representation $(0,1)$ with weights $(1,0), (1,1), (0,1)$, respectively. We eliminate $\xi^*$ of weight $(0,1)$ by means of the syzygy $\eta \eta^* + \xi^* \xi + \xi \xi^* = 0$ (the scalar $\eta \eta^* + \xi^* \xi + \xi \xi^*$ never appears in these states). The action of $R_1 R_2$ is to permute $\eta \xi \xi^* \eta \eta^*$ cyclically. Thus (A2) becomes

$$R_1 R_2 |x\rangle = (\xi \xi^* \eta^*)^y (-\eta \eta^* \xi \eta \eta^* \xi^*)^{y-x} (\eta \eta^* \xi)^{(p-q)/3} \times (\eta \eta^*)^\alpha (q-x)! \sum_{a=0}^{\infty} (\xi \xi^* \eta) ^{\alpha-a} \alpha! (q-x)! \alpha! (q-x)!.$$  \hspace{1cm} (A3)

The contribution of $|x\rangle$ to the trace is the coefficient of $|x\rangle$ on the right-hand side of (A3) and the complete trace is hence the left-hand side of (A1).
11A. Speiser, Die theorie der Gruppen von endlichen Ordnung (Vollständigen Gruppen, Birkhäuser, 1956), §42.

15Equations (4.7) and (5.1) are the first published generating functions for multiplicities of dominant weights. They were derived using methods described fully in R. Gaskell, J. Patera, and R. T. Sharp, Generating Functions in Group Representation Theory (in preparation).