Chaos in Classical and Quantum Cosmological Billiards

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Friedmann Cosmological Models

1922 A. Friedmann: first, exact non stationary cosmological solutions with matter

$$ds^2 = -dt^2 + a^2(t)\, d\ell^2_{(k)} \leftarrow \text{space of curvature } k\text{; e.g. } S_3 \text{ for } k = +1$$

Friedmann equations for the scale factor $a(t)$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3p)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\,\rho$$

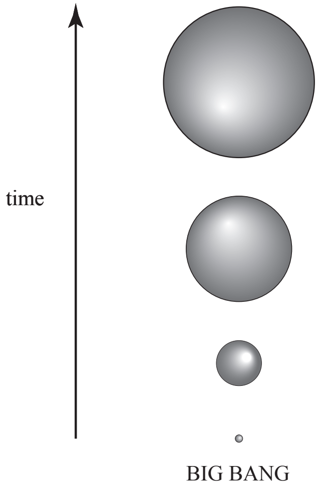
Cosmological Singularities?

If $p = w\rho$ (and $\Lambda = 0$), $\exists$ singularity where $a(t) \to 0$

near $t \to 0$ $\quad a(t) \propto t^{\frac{2}{3(1+w)}}$

e.g. $a(t) \propto t^{1/2}$ for radiation-dominated universe $\left(w = \frac{1}{3}\right)$

Cosmological Singularity in Friedmann Universe



Genericity of Cosmological Singularities?

Landau 1959: Is the big bang singularity of Friedmann universes a generic property of general relativistic cosmologies, or is it an artefact of the high degree of symmetry of these solutions?

Khalatnikov and Lifshitz 1963: look for generic inhomogeneous and anisotropic solution near a singularity

$$ds^2 = -dt^2 + (a^2\,\ell_i\,\ell_j + b^2\,m_i\,m_j + c^2\,n_i\,n_j)dx^i\,dx^j$$

single Friedmann scale factor $a(t) \to$ three inhomogeneous scale factors $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$

KL63 did not succeed in finding the "general" solution of the complicated, coupled dynamics of a, b, c and tentatively concluded that a singularity is not generic.

Genericity of Cosmological Singularities?

Hawking 1967, Hawking-Penrose 1970: Theorems about genericity of cosmological "singularity".

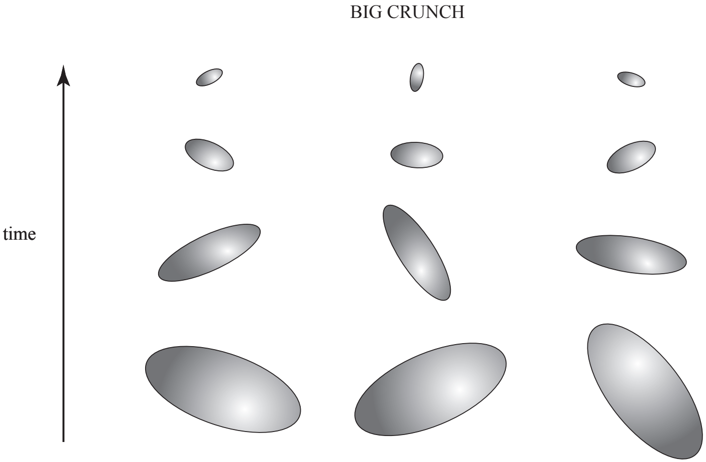
However, they prove generic "incompleteness" of spacetime, without giving any information about the "singularity".

Belinsky, Khalatnikov, Lifshitz 1969:

• construct the "general" solution near $abc \to 0$ of the coupled (inhomogeneous) dynamics of $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$,

• find that, at each point of space $\mathbf{x}$, the dynamics of a, b, c is chaotic.

BKL chaos near a big bang or a big crunch



Cosmological Billiards

(Misner, Chitre, Damour-Henneaux-Nicolai)

$$ds^2 = -dt^2 + (a^2\,\ell_i\,\ell_j + b^2\,m_i\,m_j + c^2\,n_i\,n_j)dx^i\,dx^j$$

exponential parametrisation: $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$

Lagrangian ruling the dynamics of the β's at each spatial point

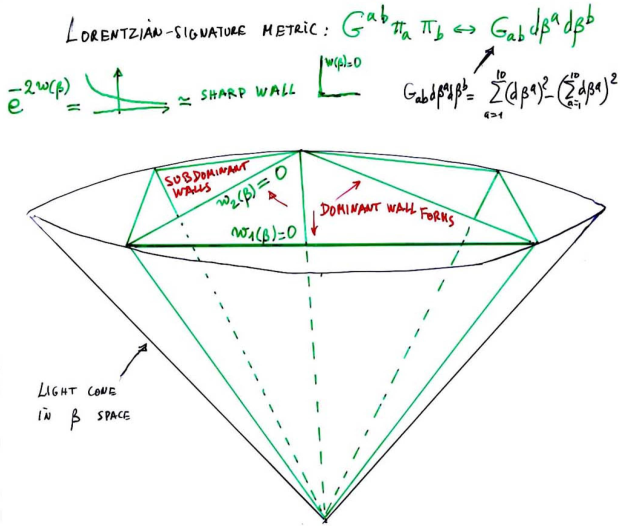
$$\mathcal{L} = \frac{1}{2}\,G_{ab}\,\dot\beta^a\,\dot\beta^b - V(\beta)$$

Kinetic metric $G_{ab}\,\dot\beta^a\,\dot\beta^b = \sum_a (\dot\beta^a)^2 - \left(\sum_a \dot\beta^a\right)^2$

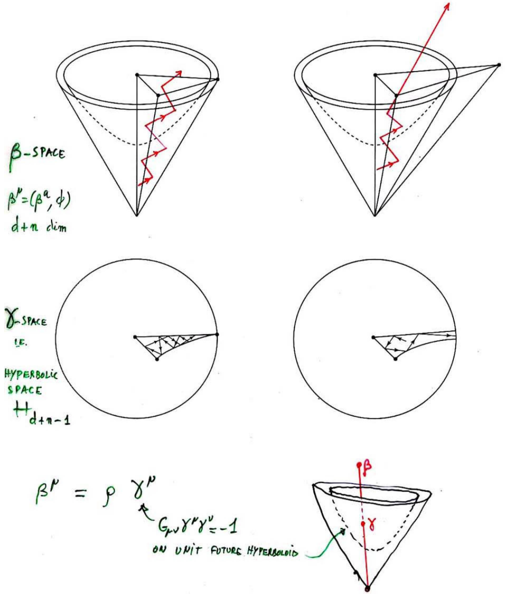
Potential $V(\beta) = \sum_a c_A(\dots)\,e^{-2w_A(\beta)}$

Wall forms $w_A(\beta)$: e.g. gravitational walls: $w^{(g)}_{abc}(\beta) = \sum_e \beta^e + \beta^a - \beta^b - \beta^c$

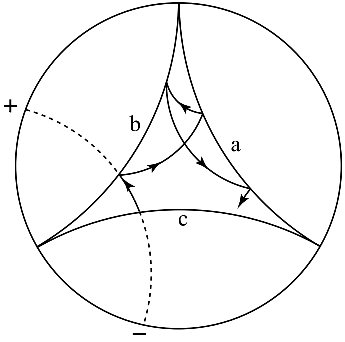
Billiard in β space

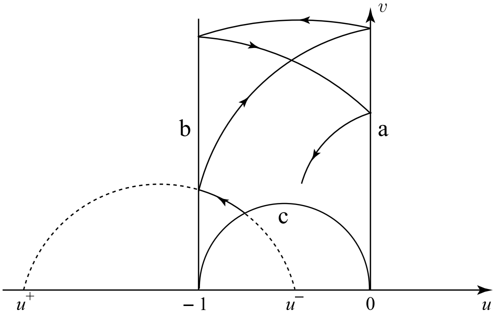


Einstein Billiards (chaotic versus non-chaotic)



Chaotic billiard for $D = 4$ gravity (BKL, Misner, Chitre)





Hopscotch dynamics in the (u^+, u^-) plane

(Damour-Lecian 2011)

Each collision on a wall a, b or c induces an homographic transformation of u^+ and u^-:

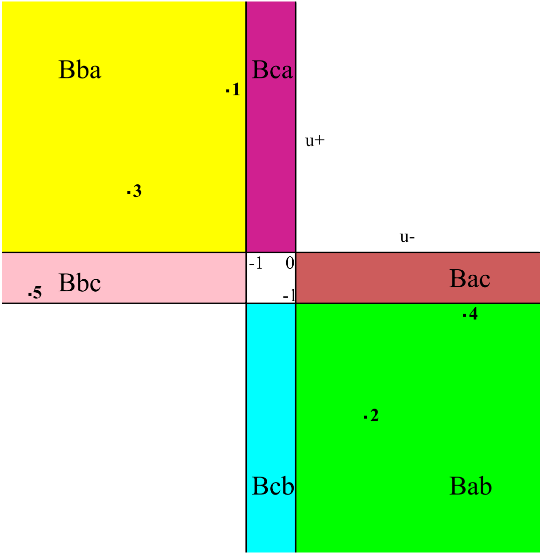
$$u^\pm = \frac{\alpha\, u^\pm + \beta}{\gamma\, u^\pm + \delta} \quad \text{with integral matrix } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ of determinant } -1$$

wall $a \leftrightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; wall $b \leftrightarrow B = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$; wall $c \leftrightarrow C = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$

Statistical properties of BKL billiard (Lifshitz, Lifshitz, Khalatnikov 1971; Lifshitz, Khalatnikov, Sinai, Khanin, Shchur 1983; Chernoff, Barrow 1983; Kirillov, Montani 1997; Damour, Lecian 2011)

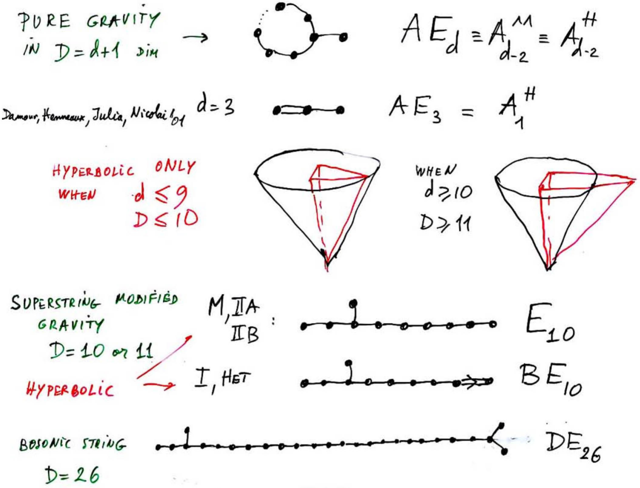
$\exists$ invariant measure in (u^+, u^-) plane: $\mu = \frac{du^+ \wedge du^-}{(u^+ - u^-)^2}$

Hopscotch dynamics in the (u^+, u^-) plane



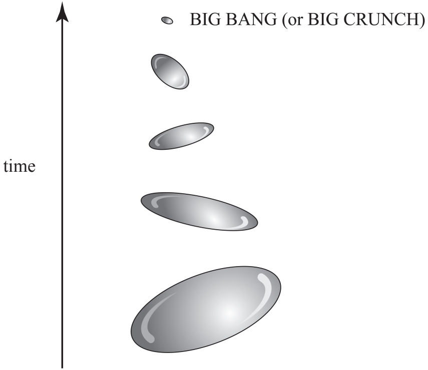
Cosmological Singularities and Hyperbolic Kac-Moody Algebras

Damour, Henneaux 2001; Damour, Henneaux, Julia, Nicolai 2001; Damour, Henneaux, Nicolai 2002



Quantum Supersymmetric Billiards

• Quantum dynamics of a supersymmetric triaxially squashed three-sphere



Supersymmetric triaxially squashed three-sphere

(Damour, Spindel 2013)

Technically: Reduction to one, time-like, dimension of the action of $D = 4$ simple supergravity for an $SO(3)$-homogeneous (Bianchi IX) cosmological model

$$
\begin{aligned}
g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} &= -N^2(t) dt^2 \\
&+ g_{ab}(t)(\tau^a(x) + N^a(t) dt)(\tau^b(x) + N^b(t) dt)\,,
\end{aligned}
$$

τ^a: left-invariant one-forms on $SU(2) \approx S_3 : d\tau^a = \frac{1}{2}\, \varepsilon_{abc}\, \tau^b \wedge \tau^c$

Dynamical degrees of freedom

- Gauss-decomposition of the metric:

$$g_{bc} = \sum_{\widehat{a}=1}^{3} e^{-2\beta^a} \, S^{\widehat{a}}{}_b(\varphi_1, \varphi_2, \varphi_3) \, S^{\widehat{a}}{}_c(\varphi_1, \varphi_2, \varphi_3)$$

six metric dof:

$$\begin{aligned} \beta^a &= (\beta^1(t), \beta^2(t), \beta^3(t)) \\ &= \text{cologarithms of the squashing parameters } a, b, c \text{ of 3-sphere} \end{aligned}$$

and three Euler angles:

$$\varphi_a = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$$

- 3×4 gravitino components Φ^a_A, $a = 1, 2, 3$; $A = 1, 2, 3, 4$.

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\theta^a} \, \dot{\theta}^a + \frac{i}{2} \, G_{ab} \, \Phi_A^a \, \dot{\Phi}_A^b + \bar{\Psi}_{\hat{0}}^{\prime A} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

G_{ab}: Lorentzian-signature quadratic form:

$$G_{ab} \, d\beta^a \, d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a's:

$$\frac{1}{2} \, G_{ab} \, \dot{\beta}^a \, \dot{\beta}^b$$

Lagrange multipliers $\longrightarrow$ Constraints $\mathcal{S}_A \approx 0$, $H \approx 0$, $H_a \approx 0$

Quantization

• Bosonic dof:

$$\widehat{\pi}_a = -i \frac{\partial}{\partial \beta^a} \, ; \quad \widehat{p}_{\theta^a} = -i \frac{\partial}{\partial \theta^a}$$

• Fermionic dof:

$$\widehat{\Phi}_A^a \, \widehat{\Phi}_B^b + \widehat{\Phi}_B^b \, \widehat{\Phi}_A^a = G^{ab} \, \delta_{AB}$$

This is the Clifford algebra $\mathrm{Spin}\,(8^+, 4^-)$

• The wave function of the universe $\Psi_\sigma(\beta^a, \theta^a)$ is a 64-dimensional spinor of $\mathrm{Spin}\,(8, 4)$ and the gravitino operators Φ_A^a are 64×64 "gamma matrices" acting on Ψ_σ, $\sigma = 1, \ldots, 64$

Dirac Quantization of the Constraints

$$\widehat{\mathcal{S}}_A \Psi = 0\,, \quad \widehat{H}\Psi = 0\,, \quad \widehat{H}_a \Psi = 0$$

Diffeomorphism constraint $\Leftrightarrow \widehat{p}_{\theta^a}\Psi = -i\,\frac{\partial}{\partial\theta^a}\Psi = 0$

$\longrightarrow$ Wave function $\Psi(\beta^a)$ submitted to constraints

$$\widehat{\mathcal{S}}_A(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0\,, \quad \widehat{H}(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0$$

$\widehat{\pi}_a = -i\,\frac{\partial}{\partial\beta^a} \Rightarrow 4 \times 64 + 64$ PDE's for the 64 functions $\Psi_\sigma(\beta^1, \beta^2, \beta^3)$

Heavily overdetermined system of PDE's

(Open) Superalgebra satisfied by the $\widehat{\mathcal{S}}_A$'s and $\widehat{H}$

$$\widehat{\mathcal{S}}_A = \frac{i}{2}\, \Phi_A^a\, \frac{\partial}{\partial \beta^a} + \ldots$$

$$\widehat{H} = -\frac{1}{2}\, G^{ab}\, \frac{\partial}{\partial \beta^a}\, \frac{\partial}{\partial \beta^b} + \text{spin-dependent potential terms}$$

$$\widehat{\mathcal{S}}_A\, \widehat{\mathcal{S}}_B + \widehat{\mathcal{S}}_B\, \widehat{\mathcal{S}}_A = 4\, i \sum_C \widehat{L}_{AB}^C(\beta)\, \widehat{\mathcal{S}}_C + \frac{1}{2}\, \widehat{H} \delta_{AB}$$

$$[\widehat{\mathcal{S}}_A, \widehat{H}] = \widehat{M}_A^B\, \widehat{\mathcal{S}}_B + \widehat{N}_A \widehat{H}$$

Solutions of SUSY constraints

Overdetermined system of 4×64 Dirac-like equations

$$\widehat{\mathcal{S}}_A \Psi = \left(\frac{i}{2} \, \Phi_A^a \, \frac{\partial}{\partial \beta^a} + \dots \right) \Psi = 0$$

Space of solutions is a mixture of "discrete states" and "continuous states", depending on fermion number $N_F = C_F - 3$. $\exists$ solutions for both even and odd N_F.

$\exists$ continuous states (parametrized by initial data comprising arbitrary *functions*) at $C_F = -1, 0, +1$.

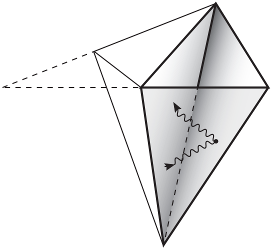
Our results complete/correct inconclusive studies started long ago: D'Eath 93, D'Eath-Hawking-Obregon 93, Csordas-Graham 95, Obregon 98, ...

Quantum Supersymmetric Billiard

The spinorial wave function of the universe $\Psi(\beta^a)$ propagates within the (various) Weyl chamber(s) and "reflects" on the walls ($=$ simple roots of AE_3). In the small-wavelength limit, the "reflection operators" define a spinorial extension of the Weyl group of AE_3 (Damour Hillmann 09) defined within some subspaces of $\mathrm{Spin}(8,4)$

$$\widehat{\mathcal{R}}_{\alpha_i} = \exp\left(-i\,\frac{\pi}{2}\,\widehat{\varepsilon}_{\alpha_i}\,\widehat{J}_{\alpha_i}\right)$$

with $\widehat{J}_{\alpha_i} = \{\widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}\}$ and $\widehat{\varepsilon}^2_{\alpha_i} = \mathrm{Id}$



Conclusions

• The BKL conjecture about the chaotic behaviour of the generic solution of Einstein's equations near a cosmological singularity has been confirmed and extended.

• The dynamics of pure gravity $(g_{\mu\nu})$ is chaotic in spacetime dimensions $D \leq 10$ and non-chaotic in dimensions $D \geq 11$; but the extended gravity models obtained from the low-energy limit of superstring theory are chaotic.

• The chaotic behaviour of the anisotropic scale factors of the metric is described by a billiard in an auxiliary (β) Lorentzian space. This billiard happens to coincide with the Weyl chamber of an hyperbolic Kac-Moody algebra. [This corresponds to a special "arithmetic chaos".]

•There is a corresponding chaotic behaviour of the fermionic degrees of freedom.

•This chaos is linked to a hidden symmetry: e.g. E_{10} for M-theory. This link suggests a new (holographic) scenario for the (de-)emergence of space at a cosmological singularity.